广西科学院学报 2015,31(1):59～63
Journal of Guangxi Academy of Sciences Vol. 31, No. 1 February 2015

网络优先数字出版时间:2015-01-14
网络优先数字出版地址:http://www.cnki.net/kcms/detail/45.1075.N.20150114.1035.011.html

On the Set-coloring Vertex and Edge Folkman Numbers*

集染色顶点和集染色边的 Folkman 数

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Abstract: Given a simple graph G and positive integers $a_1, a_2, \cdots, a_k$, we write $G \to (a_1, a_2, \cdots, a_k)_r^v$ (resp. $G \to (a_1, a_2, \cdots, a_k)_r^e$) if for any k-coloring of $V(G)$ (resp. $E(G)$) in which each vertex (edge) is colored with an r-subset of $\{1, \cdots, k\}$. There exists a complete subgraph of order a_i in which every vertex (resp. edge) is colored with an r-subset containing color i for some $i \in \{1, \cdots, k\}$. In this paper, for integer $t > \max\{a_1, a_2, \cdots, a_k\}$, the set-coloring vertex (resp. edge) Folkman number is defined and studied, $F_v^{(r)}(a_1, a_2, \cdots, a_k; t) = \min\{|V(G)|: G \to (a_1, a_2, \cdots, a_k)_r^v$ and $K_t \nsubseteq G\}$ (resp. $F_e^{(r)}(a_1, a_2, \cdots, a_k; t) = \min\{|V(G)|: G \to (a_1, a_2, \cdots, a_k)_r^e$ and $K_t \nsubseteq G\}$.)

Key words: Folkman number, set coloring, Ramsey theory

摘要:对于给定的简单图 G 和正整数 $a_1, a_2, \cdots, a_k$, $G \to (a_1, a_2, \cdots, a_k)_r^v$ ($G \to (a_1, a_2, \cdots, a_k)_r^e$) 是指,对于 $V(G)$ ($E(G)$) 的任意 k-染色,其中每个顶点(边)被用 $\{1, \cdots, k\}$ 的一个 r-子集来染色,存在 $i \in \{1, \cdots, k\}$ 和一个阶为 a_i 的完全子图,其中每个顶点(边)被一个包含颜色 i 的 r-子集染色.本文在整数 $t > \max\{a_1, a_2, \cdots, a_k\}$ 的条件下,定义并研究下述集染色顶点(边)Folkman 数: $F_v^{(r)}(a_1, a_2, \cdots, a_k; t) = \min\{|V(G)|: G \to (a_1, a_2, \cdots, a_k)_r^v$ 且 $K_t \nsubseteq G\}$ (类似地, $F_e^{(r)}(a_1, a_2, \cdots, a_k; t) = \min\{|V(G)|: G \to (a_1, a_2, \cdots, a_k)_r^e$ 且 $K_t \nsubseteq G\}$).

关键词:Folkman 数 集染色 Ramsey 理论
中图分类号:O175 **文献标识码**:A **文章编号**:1002-7378(2015)01-0059-05

0 Introduction

Let G be a finite simple graph that contains no loops or multiple edges. Denote by $V(G)$ the set of its vertices and $E(G)$ the set of its edges. A graph that may contain multiple edges but does not contain loops is called a multigraph. We only consider

收稿日期:2014-11-10
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* 国家自然科学基金项目(批准号 11361008,61309015)和广西自然科学基金项目(2011GXNSFA018142)资助。

graphs without loops in this paper. A cycle of order n is denoted by C_n . A complete graph of order n is denoted by K_n . The clique number of G is denoted by $cl(G)$. A (s,t) graph is a graph that contains neither an s-clique nor a t-independent set. The set $\{1,2,\cdots,k\}$ is denoted by $[k]$.

Throughout this paper, supposing $a_1, a_2, \cdots, a_k, a_{k+1}, a, t$ and r are positive integers, where $1 \leqslant r < k$ and $a_i \geqslant 2$ for any $i \in [k]$.

We write $G \to (a_1, a_2, \cdots, a_k)_r^v$ (resp. $G \to (a_1, a_2, \cdots, a_k)_r^e$) if for every k-coloring of $V(G)$ (resp. $E(G)$) in which each vertex (edge) is colored with an r-subset of $[k]$, there exists a complete subgraph of order a_i in which every vertex (resp. edge) is colored with an r-subset of $[k]$ containing color i for some $i \in [k]$.

In most coloring problems in graph theory, we often color a vertex or an edge with one and only one color. Such a coloring was generalized by some graph theorists, such that every edge or vertex is not mapped to one color, but a subset of the given color set $[k]$.

In reference[1], Harary et al studied Ramsey numbers for multigraph. They said that it appears there are no new interesting Ramsey numbers for multigraphs. They obtained such a conclusion because that they admitted the edges between the same pair of vertices colored with same colors, rather than used the idea of set-colorings.

In reference[2], the multigraph Ramsey number $f^{(r)}(a_1, a_2, \cdots, a_k)$ was defined to be the minimum positive integer n such that $K_n \to (a_1, a_2, \cdots, a_k)_r^e$. Note that set-coloring of edges in a graph is the same as the edge-coloring of a related multigraph, in which the edges between the same pair of vertices must be in different colors. So $f^{(r)}(a_1, a_2, \cdots, a_k)$ is the Ramsey number for multigraphs in which each pair of vertices are joined by r edges. It is a generalization of the classic Ramsey number, because that the classic Ramsey number $R(a_1, a_2, \cdots, a_k) = f^{(1)}(a_1, a_2, \cdots, a_k)$.

Now let us give an example to tell why we think multigraph Ramsey numbers, i. e. the set-coloring generalization of classical Ramsey numbers, may be interesting and important. In reference[2], it was proved that if $k > r \geqslant 1, q \geqslant 3$, then

$$f_k^{(r)}(q) \geqslant (\frac{k}{r})^{\frac{q-1}{2}} (\frac{q!}{k})^{\frac{1}{q}}.$$

So

$$f_3^{(2)}(q) \geqslant (\frac{3}{2})^{\frac{q-1}{2}} (\frac{q!}{3})^{\frac{1}{q}}.$$

We know that $f_3^{(2)}(q) \leqslant R(q,q)$, and the former may be much smaller than the later. But on the other hand, we do know how to give $R(q,q)$ such a lower bound now.

For positive integer $t > \max\{a_1, a_2, \cdots, a_k\}$, let

$F_v^{(r)}(a_1, a_2, \cdots, a_k; t) = \{G: G \to (a_1, a_2, \cdots, a_k)_r^v$ and $cl(G) < t\}$, and

$F_e^{(r)}(a_1, a_2, \cdots, a_k; t) = \{G: G \to (a_1, a_2, \cdots, a_k)_r^e$ and $cl(G) < t\}$.

The set-coloring vertex Folkman number is defined as

$F_v^{(r)}(a_1, a_2, \cdots, a_k; t) = \min\{ |V(G)| : G \in F_v^{(r)}(a_1, a_2, \cdots, a_k; t) \}$,

and the set-coloring edge Folkman number is defined as

$F_e^{(r)}(a_1, a_2, \cdots a_k; t) = \min\{ |V(G)| : G \in F_e^{(r)}(a_1, a_2, \cdots, a_k; t) \}$.

If $a_1 = a_2 = \cdots = a_k = a$, we denote $F_v^{(r)}(a_1, a_2, \cdots, a_k; t)$ (resp. $F_e^{(r)}(a_1, a_2, \cdots, a_k; t)$) as $F_{vk}^{(r)}(a;t)$ (resp. $F_{ek}^{(r)}(a;t)$), and $F_v^{(r)}(a_1, a_2, \cdots, a_k; t)$ (resp. $F_e^{(r)}(a_1, a_2, \cdots, a_k; t)$) as $F_{vk}^{(r)}(a;t)$ (resp. $F_{ek}^{(r)}(a;t)$).

Thus vertex Folkman number $F_v(a_1, a_2, \cdots, a_k; t) = F_v^{(1)}(a_1, a_2, \cdots, a_k; t)$ and edge Folkman number $F_e(a_1, a_2, \cdots, a_k; t) = F_e^{(1)}(a_1, a_2, \cdots, a_k; t)$. In 1970, Folkman[3] proved that for any integer $t > \max\{a_1, a_2, \cdots, a_k\}$, both $F_v(a_1, a_2, \cdots, a_k; t)$ and $F_e(a_1, a_2; t)$ were nonempty. His result was generalized to $F_e(a_1, a_2, \cdots, a_k; t)$ for arbitrary $k \geqslant 2$ by Nešetřil et al[4].

There are many new problems to be studied, in which some need much computation. We wish that these set-coloring generalizations of vertex Folkman numbers and edge Folkman numbers will be interesting for mathematicians and computer scientists.

The rest of this paper is organized as follows. In Section 1 some basic results are proved. In Section 2, the upper bound for set-coloring vertex Folkman number is studied. In Section 3 a result between the set-coloring edge Folkman number and the set-coloring vertex Folkman number is proved. The value of $F_e^{(2)}(3,3,3;4)$ is decided in Section 4.

1 Some basic results on set-coloring Folkman numbers

In this section we will give some basic results on set-coloring vertex and edge Folkman numbers, most without proofs, because that they can be proved similar to some known results on either multigraph Ramsey numbers or vertex and edge Folkman numbers.

It is not difficult to know the existence of set-coloring vertex and edge Folkman numbers. In fact, we have the following two theorems on set-coloring vertex Folkman numbers, which are similar to those on multigraph Ramsey numbers in reference[2]. It is not difficult to see that we can prove similar theorems on set-coloring edge Folkman numbers.

Theorem 1 If $r \geqslant 2$ and $t > \max\{a_1, a_2, \cdots, a_{k+1}\}$, then

$$F_v^{(r)}(a_1, a_2, \cdots, a_k; t) \leqslant F_v^{(r)}(a_1, a_2, \cdots, a_{k+1}; t) \leqslant F_v^{(r-1)}(a_1, a_2, \cdots, a_k; t).$$

Theorem 2 If r, k, a, t, s are all positive integers, and $t > a \geqslant 2$, then $F_{vk}^{(r)}(a;t) \leqslant F_{vsk}^{(sr)}(a;t)$.

We need only to prove that $F_{vsk}^{(sr)}(a;t) \subseteq F_{vk}^{(r)}(a;t)$. Let $G \in F_{vsk}^{(sr)}(a;t)$. If $G \notin F_{vk}^{(r)}(a;t)$, there is a coloring of $V(G)$ in which each vertex v is colored with an r-subset $C[v]$ of $[k]$, such that there doesn't exist any monochromatic K_a in color i for some $i \in [k]$. Based on this set-coloring method, we color each vertex $v \in V(G)$ with $\{j \mid j \in [sk], j \equiv i (\bmod k)$ for some $i \in C[v]\}$. So $F_{vsk}^{(sr)}(a;t) \subseteq F_{vk}^{(r)}(a;t)$. $F_{esk}^{(sr)}(a;t) \subseteq F_{ek}^{(r)}(a;t)$ can be proved similarly.

The off-diagonal case of Theorem 2 can be obtained similarly. For instance, we have $13 = F_v(3,4;5) \leqslant F_v^{(2)}(3,3,4,4;5)$.

The following theorem is similar to the result for $r=1$ on vertex Folkman numbers.

Theorem 3 If b_1, b_2, t_1, t_2 are integers, $b_1 \leqslant t_1$ and $b_2 \leqslant t_2$, then

$$F_v^{(r)}(a_1, a_2, \cdots, a_k, b_1 + b_2; t_1 + t_2 + 1) \leqslant F_v^{(r)}(a_1, a_2, \cdots, a_k, b_1; t_1 + 1) + F_v^{(r)}(a_1, a_2, \cdots, a_k, b_2; t_2 + 1).$$

The following theorem is a generalization of a theorem in reference[5], and can be proved similarly.

Theorem 4 If $r \geqslant 1, a_1, a_2, \cdots, a_k \geqslant 2, b_1, b_2, \cdots, b_k \geqslant 2$ and $s \geqslant \max\{a_1, a_2, \cdots, a_k\}, t \geqslant \max\{b_1, b_2, \cdots, b_k\}$, then

$$F_v^{(r)}(a_1 b_1, a_2 b_2, \cdots, a_k b_k; st + 1) \leqslant F_v(a_1, a_2, \cdots, a_k; s+1) F_v^{(r)}(b_1, b_2, \cdots, b_k; t+1).$$

2 The upper bound for set-coloring vertex Folkman number

In this section we will prove a theorem on the upper bound for the set-coloring vertex Folkman number, by which we can obtain the upper bounds for some set-coloring vertex Folkman numbers based on known upper bounds on related vertex Folkman numbers.

Theorem 5 Given a K_t-free graph G of order n, suppose the order of the maximum K_{a_i}-free induced subgraph of G is $f_i(G)$. If $\sum_{i=1}^{k} f_i(G) < m$, then $F_v^{(r)}(a_1, a_2, \cdots, a_k; t) \leqslant n$.

Proof If $F_v^{(r)}(a_1, a_2, \cdots, a_k; t) > n$, then G has $k K_{a_i}$-free induced subgraphs G_i such that any vertex among G is in just r ones of G_i, where $1 \leqslant i \leqslant k$. Thus $\sum_{i=1}^{k} |V(G_i)| = m$. Since G_i is a K_{a_i}-free induced subgraph of G, $|V(G_i)| \leqslant f_i(G)$. Therefore $\sum_{i=1}^{k} f_i(G) \geqslant \sum_{i=1}^{k} |V(G_i)| = m$. Thus if $\sum_{i=1}^{k} f_i(G) < m$, then $F_v^{(r)}(a_1, a_2, \cdots, a_k; t) \leqslant n$.

By Theorem 5, we can see that for a given K_t-free graph G of order n, if the order of the maximum K_a-free induced subgraph of G is $f(G)$, and $kf(G) < m$, then $F_{vk}^{(r)}(a;t) \leqslant n$. For instance, if $(s-1)k < m$ and $n < R(t,s)$, then $F_{vk}^{(r)}(2;t) \leqslant n$.

Suppose $f_{s,t}(n) = \min\{\max\{|S| : S \in V(H)$

and $H[S]$ contains no $K_s\}$, where the minimum is taken over all K_t-free graphs H of order n (see reference[6]). Then we have the following corollary.

Corollary 1 If $kf_{a,t}(n) < n$, then $F_{vk}^{(r)}(a;t) \leqslant n$.

In reference[7], it was proved that for every integer $s \geqslant 2$, there is a positive constant $c=c(s)$ so that for every integer $n, f_{s,s+1}(n) \leqslant cn^{2/3}$. Based on this result, reference [7] proved the following statement for $r=1$, when the asymptotic is taken in k.

Corollary 2 For every integer s, there is a positive constant $c=c(s)$ such that for every integer $k, F_{vk}^{(r)}(s;s+1) \leqslant ck^3$.

It is often difficult to determine the value of $F_{vk}^{(r)}(t;t+1)$, even for $t=2, r=1$ and general k. In the following theorem we decide the values of some set-coloring vertex Folkman numbers in a simple case.

Theorem 6 If $(k-r)t < k$, then $F_{vk}^{(r)}(t;t+1) = t$.

Proof Since that $F_{vk}^{(r)}(t;t+1) \geqslant t$, so we need only prove $F_{vk}^{(r)}(t;t+1) \leqslant t$. If for any coloring of $V(K_t)$ in which each vertex is colored with an r-subset of $[k]$, there doesn't exist K_t in which every vertex is colored with i for some $i \in [k]$, then the amount of vertex colored by any color in $[k]$ is not more than $t-1$. If $k(t-1) < tr$, we can see that there exists K_t in which every vertex is colored with an r-subset of $[k]$ containing color i for some $i \in [k]$. If $(k-r)t < k$, then $k(t-1) < tr$ and $F_{vk}^{(r)}(t;t+1) \leqslant t$.

3 A result between set-coloring edge and vertex Folkman numbers

An inequality on the set-coloring vertex Folkman number and the set-coloring edge Folkman number is given in the following theorem, which may be considered as a generalization of the case $r=1$ for the vertex Folkman number and the edge Folkman number.

Theorem 7 Let $R_i = f^{(r)}(a_1, a_2, \cdots, a_{i-1}, a_i - 1, a_{i+1}, \cdots, a_k)$ for every $i \in [k]$, if $\max\{R_1, R_2, \cdots, R_k\} + 1 < t$, then $F_e^{(r)}(a_1, a_2, \cdots, a_k; t) \leqslant F_v^{(r)}(R_1, R_2, \cdots, R_k; t-1) + 1$.

Proof Let $G \in F_v^{(r)}(R_1, R_2, \cdots, R_k; t-1)$, and C be any coloring of the edges in $G+v$ with r-subset of $[k]$. We color each vertex $x \in G$ with the color set of the edge (v,x) in C. Then there exists $i \in [k]$ such that G contains a complete subgraph K_{R_i} with all the vertices in color i. If K_{R_i} contains K_{a_i-1} with all edges in color i, then $K_{a_i-1}+v$ contains a complete graph K_{a_i} with all edges incolor i. If it does not, then there exists j different from i such that K_{R_j} contains K_{a_j} with all edges in color j. Thus, any coloring of the edges in $G+v$ with r-subset of $[k]$ must result in a complete subgraph K_{a_i} of which all edges are colored with r-subset of $[k]$ that contains i for some $i \in [k]$.

4 The value of a small set-coloring edge Folkman number

By Theorem 1, we have $F_e^{(2)}(3,3,3;4) \leqslant F_e(3,3;4)$. In this section we will decide the value of $F_e^{(2)}(3,3,3;4)$. It is not difficult to have lemma 1 which will be used in the proof of Theorem 8. Lemma 1 is a generalization of the known result on vertex and edge Folkman numbers, i. e. the case $r=1$.

Lemma 1 For integer $t > \max\{a_1, a_2, \cdots, a_k\}$, if $G \in F_e^{(r)}(a_1, a_2, \cdots, a_k; t)$ and $H \in F_v^{(r)}(a_1, a_2, \cdots, a_k; t)$, then $\chi(G) \geqslant f^{(r)}(a_1, a_2, \cdots, a_k)$ and $r_\chi(H) \geqslant \sum_{i=1}^{k}(a_i - 1) + 1$.

Theorem 8 $F_e^{(2)}(3,3,3;4) = 11$.

Proof Let H be a K_4-free graph of order $F_e^{(2)}(3,3,3;4)$ such that $H \to (3,3,3)_2^e$. By Lemma 1 and $f_3^{(2)}(3)=5$ (see reference[2]), we have $\chi(H) \geqslant 5$. So $F_e^{(2)}(3,3,3;4) \geqslant F_v(2,2,2,2;4) = 11$.

On the other hand, a K_4-free graph of order 11 in $F_v(2,2,2,2;4)$, as shown in Figure 1, was found by Nenov and can be found in reference[8]. Let this graph be G.

Suppose the color set is $\{1,2,3\}$. We know there is a natural map f between an edge-coloring $C^{(2)}$ of G and an edge-coloring C of G as follows. Note subsets of order 2 in $\{1,2,3\}$ is used in $C^{(2)}$, and subsets of order 1 in $\{1,2,3\}$ is used in C. The edge between u and v in $C(G)$ is in the color different from the two colors used on the edge between u and v in $C^{(2)}(G)$. So if $G \in F_e^{(2)}(3,3,3;4)$, then there must be a coloring $C(G)$ in which the three edges of any K_3 in $C(G)$ are in three different colors. Now we will prove that there is no such an edge coloring $C(G)$ for the graph G in Figure 1. Otherwise, suppose there is an edge-coloring $C(G)$ in which the three edges of any K_3 in $C(G)$ are in three different colors. Thus in such a $C(G)$, the 5 edges adjacent to vertex 11 must be in three colors, we might suppose two edge are in color 1, two edge are in color 2, and one edge is in color 3 as well. We may suppose (5,11) is in color 3, (1,11) and (3,11) are in color 1, (2,11) and (4,11) are in color 2. Other cases can be considered similarly. So we have $C(1,2) = C(2,3) = C(3,4) = \{3\}$, and $C(1,5) = \{2\}, C(4,5) = \{1\}$.

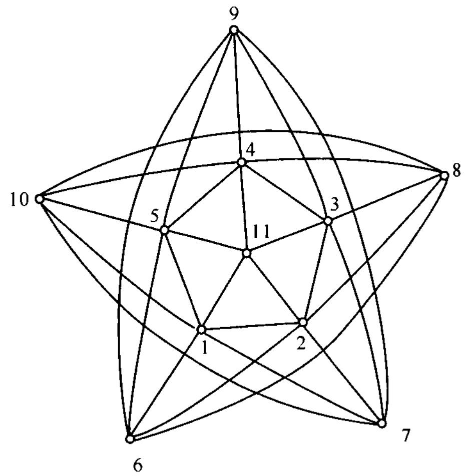

Fig. 1 A graph with chromatic number 5 without 4-cliques

$C(1,6) = \{1\}$ because it is different from $C(1,2) = \{3\}$ and $C(1,5) = \{2\}$. $C(5,6) = \{3\}$ because it is different from $C(1,5) = \{2\}$ and $C(1,6) = \{1\}$. $C(4,9) = \{2\}$ because it is different from $C(3,4) = \{3\}$ and $C(4,5) = \{1\}$. $C(5,9) = \{3\}$ because it is different from $C(4,5) = \{1\}$ and $C(4,9) = \{2\}$.

So $C(5,6) = \{3\}$ and $C(5,9) = \{3\}$, and the three edges in the subgraph of G induced by 5,6,9 can not be in three different colors, and this is a contradiction. Thus $F_e^{(2)}(3,3,3;4) \leqslant 11$.

Therefore $F_e^{(2)}(3,3,3;4) = 11$.

Acknowledgements

The authors wish to thank Professor S. P. Radziszowski for his helpful suggestions leading to the present form of this paper.

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