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一类拟七次系统原点的中心与等时中心的条件*

Centers and Isochronous Centers of a Class of Quasi-seven-degree Systems

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摘要：研究一类拟七次解析系统的中心条件与等时中心条件，得到该系统原点的前 24 个奇点量及系统原点成为中心的条件，再通过对周期常数的计算，得到其复解析系统原点成为等时中心的必要条件，并证明这些条件的充分性.

关键词：微分系统 中心 等时中心 奇点量 可积性 周期常数
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Abstract: The centers and isochronous centers of the origin for a class of quasi-seven-degree system were studied. After calculating the first twenty-four singular point values, conditions for the origin were obtained to be a center. Then necessary conditions for the origin to be an isochronous center were found through computing and simplifying the period constants. Moreover, the sufficiency were proved by effective methods.

Key words: differential equations, center, isochronous center, singular point value, integrability, period constants

在平面微分自治系统定性理论中，如何决定奇点的中心与等时中心条件是个非常困难的问题，目前许多学者对此进行了研究，并已得出大量结果[1~5]. 有关拟解析系统的报道最早见文献[6]，该类系统更多的研究可见文献[1～10]. 本文考虑如下一类拟七次微分系统的中心和等时中心条件：

$$\frac{dx}{dt}=-x+(x^2+y^2)^d(A_{70}x^7+A_{61}x^6y+A_{52}x^5y^2+A_{43}x^4y^3+A_{34}x^3y^4+A_{25}x^2y^5+A_{16}xy^6+A_{07}y^7),$$
$$\frac{dy}{dt}=y+(x^2+y^2)^d(B_{70}x^7+B_{61}x^6y+B_{52}x^5y^2+B_{43}x^4y^3+B_{34}x^3y^4+B_{25}x^2y^5+B_{16}xy^6+B_{07}y^7). \quad (1)$$

其中 $d\geqslant 1$，$A_{07}=\frac{A_{43}}{3}-\frac{9B_{16}}{7}+\frac{4B_{34}}{21}+\frac{5B_{52}}{21}$，$A_{70}=\frac{A_{34}}{3}+\frac{9B_{61}}{7}-\frac{4B_{43}}{21}-\frac{5B_{25}}{21}$，$A_{61}=\frac{A_{43}}{3}+\frac{B_{34}}{3}-\frac{B_{52}}{3}$，$A_{16}=\frac{A_{34}}{3}+\frac{B_{25}}{3}-\frac{B_{43}}{3}$，$A_{25}=A_{43}-3B_{16}+B_{34}$，$A_{52}=A_{34}-B_{43}+3B_{61}$，$B_{07}=\frac{3B_{25}}{7}+\frac{B_{43}}{7}-\frac{5B_{61}}{7}$，$B_{70}=-\frac{5B_{16}}{7}+\frac{B_{34}}{7}+\frac{3B_{52}}{7}$.

该系统在复线性变换 $z=x+iy$，$w=x-iy$，$T=it$，$i=\sqrt{-1}$ 下变为一类复平面微分系统：

$$\frac{dz}{dT}=z+(zw)^d(a_{07}w^7+a_{34}z^3w^4+a_{43}z^4w^3+a_{52}z^5w^2),$$
$$\frac{dz}{dT}=-w+(zw)^d(b_{07}z^7+b_{34}z^4w^3+b_{43}z^3w^4+b_{52}z^2w^5). \quad (2)$$

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两系统系数之间存在关系：$a_{52}=-\frac{1}{8}(3B_{16}-iB_{25}-B_{52}+3iB_{61})$，$b_{52}=\overline{a_{52}}$，

$$a_{07}=-\frac{1}{112}(3B_{16}-iB_{25}-2B_{34}+2iB_{43}+B_{52}-3iB_{61}),\ b_{07}=\overline{a_{07}},$$

$$a_{34}=\frac{1}{24}(-4iA_{34}+4A_{43}-15B_{16}+5iB_{25}+4B_{34}+4iB_{43}+5B_{52}-15iB_{61}),\ b_{34}=\overline{a_{34}},$$

$$a_{43}=\frac{1}{48}(-8iA_{34}-8A_{43}+15B_{16}-5iB_{25}-2B_{34}+2iB_{43}+5B_{52}-15iB_{61}),\ b_{43}=\overline{a_{43}}.$$

对系统(2)再做变换 $u=z^{\frac{d+7}{8}}w^{\frac{d-1}{8}}$，$v=w^{\frac{d+7}{8}}z^{\frac{d-1}{8}}$，可化为九次多项式系统：

$$\frac{du}{dT}=u+\frac{1}{8}b_{07}(1-d)u^9+\frac{1}{8}a_{07}(7+d)uv^8+\frac{1}{8}(a_{52}(7+d)+b_{34}(1-d))u^6v^3+\frac{1}{8}(a_{43}(7+d)+b_{43}(1-d))u^5v^4+\frac{1}{8}(a_{34}(7+d)+b_{52}(1-d))u^4v^5=U(u,v),\tag{3}$$

$$\frac{dv}{dT}=-v-\frac{1}{8}a_{07}(1-d)v^9-\frac{1}{8}b_{07}(7+d)vu^8-\frac{1}{8}(b_{52}(7+d)+a_{34}(1-d))v^6u^3-\frac{1}{8}(b_{43}(7+d)+a_{43}(1-d))v^5u^4-\frac{1}{8}(b_{34}(7+d)+a_{52}(1-d))v^4u^5=-V(u,v).$$

文献[11]定义了复自治系统的复中心与奇点量，文献[12]定义了复自治系统的等时中心与周期常数．他们得出结论：如果复自治系统的系数满足共轭条件且坐标原点是中心(等时中心)，则它的伴随实系统的原点也是中心(等时中心)．因此，系统(1)的中心与等时中心的必要条件可以由它的伴随复系统(3)的奇点量与周期常数得到．

1　系统原点的中心条件

由文献[13]给出的奇点量递推公式，推导并化简奇点量公式得到：

定理 1.1　系统(3)原点的前24个非恒等于零的奇点量为

$u_{4i+k}\equiv 0,k=1,2,3,i=0,1,\cdots,5.$

$u_4=\frac{1}{4}(3+d)(a_{43}-b_{43})$，$u_8=-\frac{1}{4}(3+d)(a_{34}a_{52}-b_{34}b_{52})$，

当 $a_{52}b_{52}=0$，则有 $u_{12}=u_{16}=0$，$u_{20}=-\frac{1}{512}(3+d)(7+d)(9+2d)(11+3d)I_0$，$u_{24}=-\frac{1}{2359296}(3+d)(10784285+5785128d+757483d^2)r_4I_0$．

当 $a_{52}b_{52}\neq 0$，令 $a_{34}=rb_{52}$，$b_{34}=ra_{52}$，

则 $u_{12}=u_{16}=0$，

$$u_{20}=-\frac{1}{1536}(3+d)(-5+3r)(1-d+7r+dr)(-3-2d+9r+2dr)(-7-3d+11r+3dr)J_0,$$

$$u_{24}=\frac{1}{7077888}(3+d)r_4F_0J_0,$$

以上 $r_4=\frac{1}{2}(a_{43}+b_{43})$，$I_0=-a_{34}^4b_{07}+a_{07}b_{34}^4$，$J_0=a_{07}a_{52}^4-b_{07}b_{52}^4$，

$$F_0=(907245+116540d-4387385d^2-1543920d^3+\frac{1}{a_{52}}(4b_{34}(-2001555+d(10712789+2d(3738187+578970d)))))+\frac{2}{a_{52}^2}b_{34}^2(63612363+d(74752258+d(24460643+2315880d)))+\frac{4}{a_{52}^3}b_{34}^3(45705651+d(30973051+6418906d+385980d^2))-\frac{3}{a_{52}^4}b_{34}^4(10784285+d(5785128+757483d)).$$

在上述 u_k 的表达式中，已经置 $u_1=u_2=\cdots=u_{k-1}=0,k=2,3,\cdots,24.$

定理 1.2　系统(3)原点的前24个奇点量均为零，当且仅当下列条件之一成立：

(Ⅰ) $a_{43}=b_{43},a_{52}=b_{52}=0,a_{34}^4b_{07}=a_{07}b_{34}^4,a_{07}b_{07}\neq 0.$　(4)

(Ⅱ) $a_{43}=b_{43},a_{52}=b_{52}=a_{34}=b_{34}=0.$　(5)

(Ⅲ) $a_{43}=b_{43},a_{34}a_{52}=b_{34}b_{52},a_{52}^4b_{07}=a_{07}b_{52}^4,a_{52}b_{52}\neq 0.$　(6)

(Ⅳ) $a_{43}=b_{43},a_{34}=\frac{5}{3}b_{52},b_{34}=\frac{5}{3}a_{52},a_{52}^4b_{07}=a_{07}b_{52}^4.$　(7)

(Ⅴ) $a_{43}=b_{43}=a_{07}=b_{07}=0,a_{34}=\frac{7+3d}{11+3d}b_{52},b_{34}=\frac{7+3d}{11+3d}a_{52},a_{52}b_{52}\neq 0.$　(8)

(Ⅵ) $a_{43}=b_{43}=a_{07}=b_{07}=0,a_{34}=\frac{-1+d}{7+d}b_{52},b_{34}=\frac{-1+d}{7+d}a_{52},a_{52}b_{52}\neq 0.$　(9)

(Ⅶ) $a_{43}=b_{43}=a_{07}=b_{07}=0,a_{34}=\frac{3+2d}{9+2d}b_{52},b_{34}=\frac{3+2d}{9+2d}a_{52},a_{52}b_{52}\neq 0.$　(10)

引理 1.1 系统(3)的所有Lie-不变量为

$a_{43}, b_{43}, a_{07}b_{07}, a_{34}b_{34}, a_{34}a_{52}, b_{52}b_{34}, b_{52}a_{52}, b_{07}a_{34}^4, b_{07}a_{34}^3b_{52}, b_{07}a_{34}^2b_{52}^2, b_{07}a_{34}b_{52}^3, b_{07}b_{52}^4, a_{07}b_{34}^4, a_{07}b_{34}^3a_{52}, a_{07}b_{34}^2a_{52}^2, a_{07}b_{34}a_{52}^3, a_{07}a_{52}^4$.

定理 1.3 系统(3)原点为中心的充要条件是定理1.2中7个条件之一成立.

证明 必要性是显然的,下证充分性.在定理1.2中,如果条件(Ⅰ),(Ⅱ),(Ⅲ)成立,即系统(3)右端系数满足广义对称原理的条件.

如果条件(Ⅳ)成立,系统(3)有积分因子

$$M(u,v)=u^{\frac{1-5d}{3+d}}v^{\frac{1-5d}{3+d}}. \tag{11}$$

如果条件(Ⅴ)成立,系统(3)有通积分

$$F(u,v)=\frac{1}{5u^{10}v^{10}}(-11-3d-(15+5d)u^5v^3a_{52}-(15+5d)u^3v^5b_{52}). \tag{12}$$

如果条件(Ⅵ)成立,系统(3)有通积分

$$F(u,v)=\frac{1}{5u^5v^5}(-7-d-(15+5d)u^5v^3a_{52}-(15+5d)u^3v^5b_{52}). \tag{13}$$

如果条件(Ⅶ)成立,系统(3)有积分因子

$$M(u,v)=u^{-\frac{23}{3}}v^{-\frac{23}{3}}. \tag{14}$$

2 系统原点的等时中心条件

由文献[12]给出的周期常数的递推公式,推导并化简,再根据定理1.2的中心条件,分7种情况讨论系统(3)原点的等时中心条件.

情形1 中心条件(4)成立.因为$a_{34}b_{34}\neq 0$,令

$$a_{43}=b_{43}=r43, a_{07}=sa_{34}^4, b_{07}=sb_{34}^4,$$

其中s是复常数.把上式代入文献[12]的递推公式中,得系统(3)的前5个周期常数为

$$\tau_4=2r43, \tau_8=-\frac{1}{4}a_{34}b_{34}(16+4d+(7+d)s^2a_{34}^3b_{34}^3), \tau_{12}=0,$$

$$\tau_{16}=-\frac{1}{80}a_{34}^2b_{34}^2(-5712-4328d-1025d^2-75d^3+192s^2a_{34}^3b_{34}^3),$$

$$\tau_{20}=\frac{1}{192}a_{34}^4b_{34}^4s(7+d)(9+2d)(11+3d)(28+9d).$$

因为$a_{34}b_{34}\neq 0$,由$\tau_8=0$,得到$d=\frac{-16-7a_{34}^3b_{34}^3s^2}{4+a_{34}^3b_{34}^3s^2}$,把$d$代入$\tau_{16},\tau_{20}$后得到$\tau_{16}\neq 0,\tau_{20}\neq 0$.从而系统(3)原点不是等时中心.

情形2 中心条件(5)成立.计算得

$$\tau_4=2r43, \tau_8=-\frac{1}{4}(7+d)a_{07}b_{07}, \tau_{12}=\tau_{16}=\tau_{20}=0.$$

在上述表达式中,$\tau_1=\tau_2=\cdots\tau_{16}=\tau_{20}=0$的充分必要条件是$a_{07}=b_{07}=0$.在这个条件下,系统(3)是平凡的线性系统,原点是等时中心.

情形3 中心条件(6)成立.因为$a_{52}b_{52}\neq 0$,令

$$a_{43}=b_{43}=r43, a_{34}=pb_{52}, b_{34}=pa_{52}, a_{07}=sb_{52}^4, b_{07}=sa_{52}^4.$$

其中p,s是复常数,把上式代入文献[12]的递推公式中,得系统(3)的前5个周期常数为

$$\tau_4=2r43, \tau_8=-\frac{1}{4}a_{52}b_{52}F0, \tau_{12}=0, \tau_{16}=-\frac{1}{240}a_{52}^2b_{52}^2F1, \tau_{20}=\frac{1}{192}sa_{52}^4b_{52}^4F2,$$

其中$F0=-8-4d+8p+(16+4d)p^2+(7+d)s^2a_{52}^3b_{52}^3$,

$F1=-(1+p)(45d^3(-1+p)^2(-3+5p)+d^2(-345+3149p-5879p^2+3075p^3)+4d(-280+1595p-4343p^2+3246p^3)+4(-485+1711p-3624p^2+4284p^3)+64(-25+9p^2)s^2a_{52}^3b_{52}^3)$,

$F2=(1-d+7p+dp)(-3-2d+9p+2dp)(-7-3d+11p+3dp)(5+29p+8dp)$.

因为$a_{52}b_{52}\neq 0$,当$F0=F1=0$时,$F2\neq 0$.所以在此情形下系统(3)原点不是等时中心.

情形4 中心条件(7)成立.令

$$a_{43}=b_{43}=r43, a_{07}=sb_{52}^4, b_{07}=sa_{52}^4.$$

这里s是复常数,把上式代入文献[12]的递推公式中,得到系统(3)的前5个非恒等于零的周期常数为

$$\tau_4=2r43, \tau_8=-\frac{1}{36}(7+d)a_{52}b_{52}(64+9s^2a_{52}^3b_{52}^3), \tau_{12}=0,$$

$$\tau_{16}=\frac{32}{81}(7+d)(15+d)(13+3d)a_{52}^2b_{52}^2, \tau_{20}=-\frac{20}{729}(5763+1970d+155d^2)sa_{52}^4b_{52}^4.$$

在此表达式中,已置$\tau_1=\tau_2=\cdots=\tau_{16}=\tau_{20}=0$.

定理 2.1 系统(3)原点前20个周期常数为零的充要条件是下列两条件之一成立:

$$(\text{i})\ a_{43}=b_{43}=b_{34}=a_{52}=b_{07}=0, \tag{15}$$

$$(\text{ii})\ a_{43}=b_{43}=a_{34}=b_{52}=a_{07}=0. \tag{16}$$

定理 2.2 系统(3)原点是等时中心,当且仅当定理2.1的两条件之一成立.

证明 若(15)式成立,系统(3)变为

$$\frac{du}{dT}=u+\frac{1}{8}a_{07}(7+d)uv^8+\frac{1}{8}(a_{34}(7+d)+$$

$b_{52}(1-d))u^4v^5$,

$$\frac{dv}{dT}=-v+\frac{1}{8}a_{07}(-1+d)v^9+\frac{1}{8}(a_{34}(-1+d)+b_{52}(-7-d))u^3v^6. \quad (17)$$

由极坐标公式 $u=re^{i\theta}, v=re^{-i\theta}$,可得

$$\theta=\frac{1}{2i}(\log u-\log v). \quad (18)$$

沿着系统(17)的轨线,(18)式两边对 T 求导得

$$\frac{d\theta}{dT}=\frac{1}{2i}(\frac{1}{u}\frac{du}{dT}-\frac{1}{v}\frac{dv}{dT})=-\frac{1}{2}i(2+v^8a_{07}+(a_{34}+b_{52})u^3v^5). \quad (19)$$

可以找到一个 $G=-\frac{1}{16}iv^8a_{07}-\frac{1}{2}i(a_{34}+b_{52})u^3v^5$.

使得

$$\frac{dG}{dT}+\frac{d\theta}{dT}+i=0. \quad (20)$$

由文献[6]知系统(3)的原点是等时中心.

若(16)式成立,系统(3)变为

$$\frac{du}{dT}=u+\frac{1}{8}b_{07}(1-d)u^9+\frac{1}{8}(a_{52}(7+d)+b_{34}(1-d))u^6v^3,$$

$$\frac{dv}{dT}=-v+\frac{1}{8}b_{07}(-7-d)u^8v+\frac{1}{8}(a_{52}(-1+d)+b_{34}(-7-d))u^5v^4. \quad (21)$$

也可以找到一个 $G=\frac{1}{16}iu^8b_{07}+\frac{1}{2}i(a_{52}+b_{34})u^5v^3$.

所以系统(3)的原点是等时中心.

情形5　中心条件(8)成立.把(10)式代入文献[12]的递推公式中,化简得到

$$\tau_4=0,\tau_8=-\frac{12(3+d)^2a_{52}b_{52}}{(11+3d)^2},\tau_{12}=0,\tau_{16}=\frac{291(3+d)^4a_{52}^2b_{52}^2}{2(11+3d)^4},\tau_{20}=0.$$

因为 $a_{52}b_{52}\neq 0$,当 $\tau_8=0$ 时,$d<0$,所以在此情形下,系统(3)原点不是等时中心.

情形6　中心条件(9)成立.把(11)式代入文献[12]的递推公式中,计算得到

$$\tau_8=\frac{12(3+d)^2a_{52}b_{52}}{(7+d)^2},\tau_{16}=-\frac{792(3+d)^4a_{52}^2b_{52}^2}{(7+d)^4}.$$

与情形5类似,当 $\tau_8=0$ 时,$d<0$,所以在此情形下,系统(3)原点不是等时中心.

情形7　中心条件(10)成立.把(12)式代入文献[12]的递推公式中,计算得到

$$\tau_8=\frac{8(3+d)^2a_{52}b_{52}}{(9+2d)^2},\tau_{16}=-\frac{1233(3+d)^4a_{52}^2b_{52}^2}{4(9+2d)^4}.$$

与情形5类似,当 $\tau_8=0$ 时,$d<0$,所以在此情形下,系统(3)原点不是等时中心.

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