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一类三阶 KdV 方程的精确解*

Exact Solutions to a Type of Third-order KdV Equation

韦雪敏[1],朱小军[2]
WEI Xue-min[1],ZHU Xiao-jun[2]

(1. 桂林电子科技大学数学与计算科学学院,广西桂林 541004;2. 广西岑溪市归义二中,广西岑溪 543207)
(1. School of Mathematics and Computational Science, Guilin University of Electronic Technology, Guilin, Guangxi, 541004, China; 2. Guiyi Secondary School of Cenxi City in Guangxi,Cenxi,Guangxi,543207,China)

摘要:用行波变换将三阶 KdV 方程化为常微分方程,用 Riccati 方程映射法得出满足原方程的参数方程组,再结合 Mathematica 数学软件解该参数方程组,获得一类三阶 KdV 方程的精确孤立波解和周期波解.

关键词:KdV 方程 孤波解 周期波解 双曲函数正切法

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Abstract: First the third-order KdV equation is transformed into ordinary differential equation and then the parametric equations which satisfy the original equation are obtained by generalizing Riccati mapping method. The parametric equations were further solved by mathematical software Mathematica to obtain explicit solitary wave and periodic wave solutions.

Key words: KdV equation, solitary wave solutions, periodic wave solutions, hyperbolic tanh-function method

1831 年,Scott Russell 首次发现孤立子现象,之后 Gardner 等[1]利用逆散射法[2]求解了 KdV 方程,而且这种方法可以求出非线性退化方程的精确解.如今 KdV 方程的求解已经成为较活跃的研究领域之一.本文用拓展的双曲函数正切法,借助 Riccati 方程的 27 个解[3,4],再利用一种基于符号计算的代数方法,结合 Mathematical 软件,求解 KdV 方程

$$u_t + uu_x + \mu u_{xxx} = 0, \mu \neq 0. \tag{1}$$

获得若干新的显式行波解.

设

$$u(x,t) = u(\xi), \xi = kx + \omega t, \tag{2}$$

k,ω 是任意常数.把(2)式代入(1)式得

$$\omega u_\xi + kuu_\xi + \mu k^3 u_{\xi\xi\xi} = 0.$$

积分一次得

$$\omega u = -\frac{k}{2}u^2 - \mu k^3 u_{\xi\xi}. \tag{3}$$

设 $u(\xi)$ 为 $\varphi(\xi)$ 的函数,并且 $u(\xi) = \sum_{i=1}^{n} a_i \varphi^i(\xi) + a_0$,而 $\varphi(\xi)$ 满足 Riccati 方程

$$\varphi' = q\varphi^2 + p\varphi + r, \tag{4}$$

其中 p,q,r 是可变化的实常数.

由平衡方程(4)中非线性项和最高阶线性项得 $n = 2$,则

$$u(\xi) = a_0 + a_1\varphi(\xi) + a_2\varphi^2(\xi), a_2 \neq 0, \tag{5}$$

其中 a_0,a_1,a_2 是待定实常数.将(5)式代入(3)式,并利用方程(4)得到关于 $\varphi(\xi)$ 的函数.又设其各次方的系数为 0,得到方程组:

$$\begin{cases} 2a_0\omega + ka_0^2 + 2\mu k^3(a_1pr + 2a_2r^2) = 0, \\ a_1\omega + ka_0a_1 + \mu k^3(a_1p^2 + 2a_1qr + 6a_2pr) = 0, \\ 2a_2\omega + ka_1^2 + 2ka_0a_2 + 2\mu k^3(3a_1pq + 4a_2p^2 + \\ \quad 8a_2qr) = 0, \\ ka_2^2 + 6\mu k^3a_2q^2 = 0, \\ 2ka_1a_2 + 2\mu k^3(2a_1q^2 + 10a_2pq) = 0, \\ ka_2^2 + 12\mu k^3a_2q^2 = 0. \end{cases}$$

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作者简介:韦雪敏(1984-),女,硕士研究生,主要从事微分方程研究。
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求解这些代数方程得：

情形1 $a_0 = 0, a_1 = -12k^2pq\mu, a_2 = -12k^2q^2\mu, \omega = -k^3p^2\mu$.

选定不同的 p,q,r 的值，利用 Riccati 方程的解，可以得到方程(1) 新的显式精确解.

(i) $p^2 > 0, r = 0$，且 $pq \neq 0$ 有

$$\varphi_1 = -\frac{1}{2q}[p + |p|\tanh(\frac{|p|}{2}\xi)],$$

$$\varphi_2 = -\frac{1}{2q}[p + |p|\coth(\frac{|p|}{2}\xi)],$$

$$\varphi_3 = -\frac{1}{2q}[p + |p|(\tanh(|p|\xi) \pm i\operatorname{sech}(|p|\xi))],$$

$$\varphi_4 = -\frac{1}{2q}[p + |p|(\coth(|p|\xi) \pm \operatorname{csch}(|p|\xi))],$$

$$\varphi_5 = -\frac{1}{4q}[2p + |p|(\tanh(\frac{|p|}{4}\xi) \pm \coth(\frac{|p|}{4}\xi))],$$

$$\varphi_6 = \frac{1}{2q}[-p + \frac{\sqrt{p^2(A^2+B^2)} - A|P|\cosh(|p|\xi)}{A\sinh(|p|\xi) + B}],$$

$$\varphi_7 = \frac{1}{2q}[-p - \frac{\sqrt{p^2(B^2-A^2)} + A|P|\cosh(|p|\xi)}{A\sinh(|p|\xi) + B}],$$

其中 A,B 是非零的实常数，且 $B^2 - A^2 > 0$. 从而方程(1) 的孤波解为

$$u_1 = 6k^2p\mu[p + |p|\tanh(\frac{|p|}{2}\xi)] - 3k^2\mu[p + |p|\tanh(\frac{|p|}{2}\xi)]^2,$$

$$u_2 = 6k^2p\mu[p + |p|\coth(\frac{|p|}{2}\xi)] - 3k^2\mu[p + |p|\coth(\frac{|p|}{2}\xi)]^2,$$

$$u_3 = 6k^2p\mu[p + |p|(\tanh(|p|\xi) \pm i\operatorname{sech}(|p|\xi))] - 3k^2\mu[p + |p|(\tanh(|p|\xi) \pm i\operatorname{sech}(|p|\xi))]^2,$$

$$u_4 = 6k^2p\mu[p + |p|(\coth(|p|\xi) \pm \operatorname{csch}(|p|\xi))] - 3k^2\mu[p + |p|(\coth(|p|\xi) \pm \operatorname{csch}(|p|\xi))]^2,$$

$$u_5 = 3k^2p\mu[2p + |p|(\tanh(\frac{|p|}{4}\xi) \pm \coth(\frac{|p|}{4}\xi))] - \frac{3}{4}k^2\mu[2p + |p|(\tanh(\frac{|p|}{4}\xi) \pm \coth(\frac{|p|}{4}\xi))]^2,$$

$$u_6 = 6k^2p\mu[-p + \frac{\sqrt{p^2(A^2+B^2)} - A|P|\cosh(|p|\xi)}{A\sinh(|p|\xi) + B}] - 3k^2\mu[-p + \frac{\sqrt{p^2(A^2+B^2)} - A|P|\cosh(|p|\xi)}{A\sinh(|p|\xi) + B}]^2,$$

$$u_7 = 6k^2p\mu[-p - \frac{\sqrt{p^2(B^2-A^2)} + A|P|\cosh(|p|\xi)}{A\sinh(|p|\xi) + B}] - 3k^2\mu[-p + \frac{\sqrt{p^2(B^2-A^2)} + A|P|\cosh(|p|\xi)}{A\sinh(|p|\xi) + B}]^2.$$

(ii) $p^2 < 0, r = 0$，且 $pq \neq 0$，有

$$\varphi_8 = \frac{1}{2q}[-p + |p|\tan(\frac{|p|}{2}\xi)],$$

$$\varphi_9 = -\frac{1}{2q}[p + |p|\cot(\frac{|p|}{2}\xi)],$$

$$\varphi_{10} = \frac{1}{2q}[-p + |p|(\tan(|p|\xi) \pm \sec(|p|\xi))],$$

$$\varphi_{11} = -\frac{1}{2q}[p + |p|(\cot(|p|\xi) \pm \csc(|p|\xi))],$$

$$\varphi_{12} = \frac{1}{4q}[-2p + |p|(\tan(\frac{|p|}{4}\xi) - \cot(\frac{|p|}{4}\xi))],$$

$$\varphi_{13} = \frac{1}{2q}[-p + \frac{\pm\sqrt{-p^2(A^2-B^2)} - A|p|\cosh(|p|\xi)}{A\sinh(|p|\xi) + B}],$$

$$\varphi_{14} = \frac{1}{2q}[-p - \frac{\pm\sqrt{-p^2(A^2-B^2)} + A|p|\cos(|p|\xi)}{A\sinh(|p|\xi) + B}],$$

其中 A,B 是非零的实常数，且 $A^2 - B^2 > 0$. 从而方程(1) 的周期解为

$$u_8 = -6k^2p\mu[-p + |p|\tan(\frac{|p|}{2}\xi)] - 3k^2\mu[-p + |p|\tan(\frac{|p|}{2}\xi)]^2,$$

$$u_9 = 6k^2p\mu[p + |p|\cot(\frac{|p|}{2}\xi)] - 3k^2\mu[p + |p|\cot(\frac{|p|}{2}\xi)]^2,$$

$$u_{10} = -6k^2p\mu[-p + |p|(\tan(|p|\xi) \pm \sec(|p|\xi))] - 3k^2\mu[-p + |p|(\tan(|p|\xi) \pm \sec(|p|\xi))]^2,$$

$$u_{11} = 6k^2p\mu[p + |p|(\cot(|p|\xi) \pm \csc(|p|\xi))] - 3k^2\mu[p + |p|(\cot(|p|\xi) \pm \csc(|p|\xi))]^2,$$

$$u_{12} = -3k^2p\mu[-2p + |p|(\tan(\frac{|p|}{4}\xi) - \cot(\frac{|p|}{4}\xi))] - \frac{3}{4}k^2\mu[-2p + |p|(\tan(\frac{|p|}{4}\xi) -$$

$\cot(\frac{|p|}{4}\xi))]^2$,

$u_{13} = -6k^2p\mu[-p +$

$\frac{\pm\sqrt{-p^2(A^2-B^2)} - A|p|\cosh(|p|\xi)}{A\sinh(|p|\xi)+B}] -$

$3k^2\mu[-p +$

$\frac{\pm\sqrt{-p^2(A^2-B^2)} - A|p|\cos(|p|\xi)}{A\sinh(|p|\xi)+B}]^2$,

$u_{14} = -6k^2p\mu[-p -$

$\frac{\pm\sqrt{-p^2(A^2-B^2)} + A|p|\cos(|p|\xi)}{A\sinh(|p|\xi)+B}] -$

$3k^2\mu[-p -$

$\frac{\pm\sqrt{-p^2(A^2-B^2)} + A|p|\cos(|p|\xi)}{A\sinh(|p|\xi)+B}]^2$.

(iii) $r = 0, pq \neq 0$ 有

$\varphi_{15} = \frac{-pd}{q[d+\cosh(p\xi)-\sinh(p\xi)]}$,

$\varphi_{16} = -\frac{p[\cosh(p\xi)+\sinh(p\xi)]}{q[d+\cosh(p\xi)+\sinh(p\xi)]}$.

从而方程(1)的孤波解为

$u_{15} = \frac{12k^2p^2\mu d}{d+\cosh(p\xi)-\sinh(p\xi)} -$

$12\frac{k^2p^2d^2\mu}{[d+\cosh(p\xi)-\sinh(p\xi)]^2}$,

$u_{16} = \frac{12k^2p^2\mu[\cosh(p\xi)+\sinh(p\xi)]}{d+\cosh(p\xi)+\sinh(p\xi)} -$

$12\frac{k^2p^2\mu[\cosh(p\xi)+\sinh(p\xi)]^2}{[d+\cosh(p\xi)+\sinh(p\xi)]^2}$,

其中 $\xi = kx - k^3p^2\mu t$，d 是任意常数.

情形 2 $a_0 = -2k^2p^2\mu, r = 0, a_1 = -12k^2pq\mu$, $a_2 = -12k^2q^2\mu, \omega = k^3p^2\mu$.

(i) $p^2 > 0, r = 0$，且 $pq \neq 0$ 有

$u_1 = -2k^2p^2\mu + 6k^2p\mu[p + |p|\tanh(\frac{|p|}{2}\xi)] -$

$3k^2\mu[p + |p|\tanh(\frac{|p|}{2}\xi)]^2$,

$u_2 = -2k^2p^2\mu + 6k^2p\mu[p + |p|\coth(\frac{|p|}{2}\xi)] -$

$3k^2\mu[p + |p|\coth(\frac{|p|}{2}\xi)]^2$,

$u_3 = -2k^2p^2\mu + 6k^2p\mu[p + |p|(\tanh(|p|\xi) \pm i\mathrm{sech}(|p|\xi))] - 3k^2\mu[p + |p|(\tanh(|p|\xi) \pm i\mathrm{sech}(|p|\xi))]^2$,

$u_4 = -2k^2p^2\mu + 6k^2p\mu[p + |p|(\coth(|p|\xi) \pm \mathrm{csch}(|p|\xi))] - 3k^2\mu[p + |p|(\coth(|p|\xi) \pm \mathrm{csch}(|p|\xi))]^2$,

$u_5 = -2k^2p^2\mu + 3k^2p\mu[2p +$

$|p|(\tanh(\frac{|p|}{4}\xi) \pm \coth(\frac{|p|}{4}\xi))] - \frac{3}{4}k^2\mu[2p +$

$|p|(\tanh(\frac{|p|}{4}\xi) \pm \coth(\frac{|p|}{4}\xi))]^2$,

$u_6 = -2k^2p^2\mu + 6k^2p\mu[-p +$

$\frac{\sqrt{p^2(A^2+B^2)} - A|p|\cosh(|p|\xi)}{A\sinh(|p|\xi)+B}] -$

$3k^2\mu[-p + \frac{\sqrt{p^2(A^2+B^2)} - A|p|\cosh(|p|\xi)}{A\sinh(|p|\xi)+B}]^2$,

$u_7 = -2k^2p^2\mu + 6k^2p\mu[-p -$

$\frac{\sqrt{p^2(B^2-A^2)} + A|p|\cosh(|p|\xi)}{A\sinh(|p|\xi)+B}] -$

$3k^2\mu[-p - \frac{\sqrt{p^2(B^2-A^2)} + A|p|\cos(|p|\xi)}{A\sinh(|p|\xi)+B}]^2$.

(ii) $p^2 < 0, r = 0$，且 $pq \neq 0$，有

$u_8 = -2k^2p^2\mu - 6k^2p\mu[-p +$

$|p|\tan(\frac{|p|}{2}\xi)] - 3k^2\mu[-p + |p|\tan(\frac{|p|}{2}\xi)]^2$,

$u_9 = -2k^2p^2\mu + 6k^2p\mu[p + |p|\cot(\frac{|p|}{2}\xi)] -$

$3k^2\mu[p + |p|\coth(\frac{|p|}{2}\xi)]^2$,

$u_{10} = -2k^2p^2\mu - 6k^2p\mu[-p +$

$|p|(\tanh(|p|\xi) \pm \sec(|p|\xi))] - 3k^2\mu[-p +$

$|p|(\tanh(|p|\xi) \pm \sec(|p|\xi))]^2$,

$u_{11} = -2k^2p^2\mu + 6k^2p\mu[p + |p|(\cot(|p|\xi) \pm \csc(|p|\xi))] - 3k^2\mu[p + |p|(\cot(|p|\xi) \pm \csc(|p|\xi))]^2$,

$u_{12} = -2k^2p^2\mu - 3k^2p\mu[-2p +$

$|p|(\tan(\frac{|p|}{4}\xi) - \cot(\frac{|p|}{4}\xi))] - \frac{3}{4}k^2\mu[-2p +$

$|p|(\tan(\frac{|p|}{4}\xi) - \cot(\frac{|p|}{4}\xi))]^2$,

$u_{13} = -2k^2p^2\mu - 6k^2p\mu[-p +$

$\frac{\pm\sqrt{-p^2(A^2-B^2)} - A|p|\cos(|p|\xi)}{A\sinh(|p|\xi)+B}] -$

$3k^2\mu[-p +$

$\frac{\pm\sqrt{-p^2(A^2-B^2)} - A|p|\cos(|p|\xi)}{A\sinh(|p|\xi)+B}]^2$,

$u_{14} = -2k^2p^2\mu - 6k^2p\mu[-p -$

$\frac{\pm\sqrt{-p^2(A^2-B^2)} + A|p|\cos(|p|\xi)}{A\sinh(|p|\xi)+B}] -$

$3k^2\mu[-p +$

$\frac{\pm\sqrt{-p^2(A^2-B^2)} + A|p|\cos(|p|\xi)}{A\sinh(|p|\xi)+B}]^2$.

(iii) $r = 0, pq \neq 0$ 有

$u_{15} = -2k^2p^2\mu + \frac{12k^2p^2\mu d}{d+\cosh(p\xi)-\sinh(p\xi)} -$

$12\frac{k^2p^2d^2\mu}{[d+\cosh(p\xi)-\sinh(p\xi)]^2}$,

$u_{16} = -2k^2p^2\mu +$

$$\frac{12k^2p^2\mu[\cosh(p\xi)+\sinh(p\xi)]}{d+\cosh(p\xi)+\sinh(p\xi)}-12\frac{k^2p^2\mu[\cosh(p\xi)+\sinh(p\xi)]^2}{[d+\cosh(p\xi)+\sinh(p\xi)]^2},$$

其中 $\xi=kx+k^3p^2\mu t$，d 是任意常数.

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$$c'_{11}=a,c'_{20}=b. \tag{20}$$

所以，由(9)式，(14)式，(19)式，(20)式可得 $p'_1=\frac{1}{p!}(-1)^p[ab+pbc-(p-1)ad]\prod_{k=0}^{p-2}(b-kd)$. 同理可得 $q'_1=\frac{1}{p!}bc(b+d)\prod_{k=0}^{p-2}(-b+kd)$，定理2.3证明完毕.

定理2.4　p:-1型Lotka-Volterra系统(2)原点处可线性化的充要条件是下列条件之一成立：

(1)$b=0$；(2)$b=d$；(3)$b=2d$；…；(p−1)$b=$(p−2)d；(p)$a=0,c=0$.

证明　必要性. 通过对 p:-1型Lotka-Volterra系统(2)第一对可线性化量的计算可知，要使系统(2)可线性化必须使其系统所有的可线性化量均为零. 若条件(1)，(2)，…，(p)成立，则系统的第一对可线性化量为零，则所有的可线性化量全为零，故必要性成立.

充分性. 若条件(1)，(2)，(3)，(4)，…，(p−1)，(p)成立，由文献[1]易知，系统(2)是可积的. 再由引理1.2可知，系统是可线性化的.

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