一类非连续函数积分和不等式中未知函数估计的证明过程*

Proof Process of Unknown Function Estimation in a Class of Discontinuous Integral Function & Inequalities

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摘要:利用数学归纳法和积分不等式技巧,给出一类具有时滞的非连续函数积分和不等式中未知函数估计的证明过程。

关键词:不等式 积分和不等式 未知函数估计
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Abstract: The mathematical induction and integral inequalities are applied so as to provide the proof process for unknown function estimation in a class of discontinuous integral function and integral inequalities methodology.

Key words: inequalities, integro-sum inequalities, estimation of unknown function

尽管多数微分方程无法求出精确解,但是人们可以利用积分不等式技巧对其解的模进行估计.这样的估计可以证实微分方程解的存在性、有界性、唯一性和稳定性等定性性质[1~7].本文受文献[1,4～8]的启发,给出一类非连续函数积分和不等式中未知函数估计的证明过程.

1 积分和不等式

1983 年 S. D. Borysenko[3]给出积分和不等式

$$V(t) \leqslant C + \int_{t_0}^{t} P(\tau)V^m(\tau)\mathrm{d}\tau + \sum_{t_0<t_i<t} \beta_i V(t_i - 0), \forall\, t \geqslant t_0, m > 0, m \neq 1 \quad (1.1)$$

中未知函数 V 的估计,其中 $t_1 < t_2 < \cdots, \lim\limits_{t\to\infty} t_i = \infty$, $C \geqslant 0, \beta_i \geqslant 0, P(t) \geqslant 0, t_i$ 是函数 $V(t)$ 的第一类不连续点.通过对这个积分和不等式中未知函数进行估计,可以推出对应脉冲微分方程的一些重要性质,这个积分和不等式是进一步研究更复杂形式脉冲微分方程与积分和不等式的基础.

2 主要结果

定理 1[3] 假设非负逐段连续函数 $V(t)$ 满足积分和不等式(1.1),那么未知函数 $V(t)$ 有估计:当 $0 < m < 1$ 时,

$$V(t) \leqslant \prod_{t_0<t_i<t}(1+\beta_i)[C^{1-m} + (1-m)\int_{t_0}^{t} P(\tau)\mathrm{d}\tau]^{1/(1-m)}, \forall\, t \geqslant t_0; \quad (2.1)$$

当 $m > 1$ 时,对满足

$$\int_{t_0}^{t} P(\tau)\mathrm{d}(\tau) < \frac{C^{1-m}}{(m-1)\prod\limits_{t_0<t_i<t}(1+\beta_i)^{m-1}} \quad (2.2)$$

的任意 $t \geqslant t_0$ 有

$$V(t) \leqslant C\prod_{t_0<t_i<t}(1+\beta_i)[1-(m-1)\cdot (C\prod_{t_0<t_i<t}(1+\beta_i))^{m-1}\int_{t_0}^{t} P(\tau)\mathrm{d}\tau]^{-1/(m-1)}. \quad (2.3)$$

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证明 当 $0<m<1$ 时，令 $I_i=[t_{i-1},t_i)$，$i=1,2,\cdots$. 考虑 $t\in I_1$，(1.1) 式变成 $V(t)\leqslant C+\int_{t_0}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau,\forall t\in I_1$. 令 $W(t)=C+\int_{t_0}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau$，显然 $W(0)=C$，$V(t)\leqslant W(t)$. 那么有

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t}=P(t)V^m(t)\leqslant P(t)W^m(t),\forall t\in I_1. \tag{2.4}$$

由(2.4)式推出

$$\frac{\mathrm{d}W(t)}{W^m(t)}\leqslant P(t)\mathrm{d}t,\forall t\in I_1. \tag{2.5}$$

(2.5)式两边关于 t 积分，得到 $W(t)\leqslant[C^{1-m}+(1-m)\int_{t_0}^{t}P(\tau)\mathrm{d}\tau]^{1/(1-m)},\forall t\in I_1$. 根据 $W(t)$ 的定义得到

$$V(t)\leqslant[C^{1-m}+(1-m)\int_{t_0}^{t}P(\tau)\mathrm{d}\tau]^{1/(1-m)},\forall t\in I_1. \tag{2.6}$$

当 $t\in I_2$ 时，由(1.1)式推出，对于任意 $t\in I_2$ 有

$$V(t)\leqslant C+\int_{t_0}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau+\beta_1V(t_1-0)=C+\int_{t_0}^{t_1}P(\tau)V^m(\tau)\mathrm{d}\tau+\beta_1V(t_1-0)+\int_{t_1}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau\leqslant C+\int_{t_0}^{t_1}P(\tau)[C^{1-m}+(1-m)\int_{t_0}^{\tau}P(s)\mathrm{d}s]^{m/(1-m)}\mathrm{d}\tau+\beta_1[C^{1-m}+(1-m)\int_{t_0}^{t_1}P(\tau)\mathrm{d}\tau]^{1/(1-m)}+\int_{t_1}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau=(1+\beta_1)[C^{1-m}+(1-m)\int_{t_0}^{t_1}P(\tau)\mathrm{d}\tau]^{1/(1-m)}+\int_{t_1}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau. \tag{2.7}$$

类似处理 $t\in I_1$ 时的方法，由(2.7)式推出

$$V(t)\leqslant\{(1+\beta_1)^{1-m}[C^{1-m}+(1-m)\int_{t_0}^{t_1}P(\tau)\mathrm{d}\tau]+(1-m)\int_{t_1}^{t}P(\tau)\mathrm{d}\tau\}^{1/(1-m)}<(1+\beta_1)[C^{1-m}+(1-m)\int_{t_0}^{t}P(\tau)\mathrm{d}\tau]^{1/(1-m)},t_1<t<t_2. \tag{2.8}$$

假设 $t_0<t<t_k$ 时，有

$$V(t)\leqslant\prod_{t_0<t_i<t}(1+\beta_i)[C^{1-m}+(1-m)\int_{t_0}^{t}P(\tau)\mathrm{d}\tau]^{1/(1-m)}. \tag{2.9}$$

当 $t\in I_{k+1}$ 时，由(1.1)式推出

$$V(t)\leqslant C+\int_{t_0}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau+\sum_{i=1}^{k}\beta_iV(t_i-0)=C+\int_{t_0}^{t_k}P(\tau)V^m(\tau)\mathrm{d}\tau+\sum_{i=1}^{k}\beta_iV(t_i-0)+\int_{t_k}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau\leqslant C+\int_{t_0}^{t_k}P(\tau)\{\prod_{t_0<t_i<\tau}(1+\beta_i)\cdot[C^{1-m}+(1-m)\int_{t_0}^{\tau}P(s)\mathrm{d}s]^{1/(1-m)}\}^m\mathrm{d}\tau+\sum_{i=1}^{k}\beta_i\cdot\prod_{j=1}^{i-1}(1+\beta_j)[C^{1-m}+(1-m)\int_{t_0}^{t_i}P(s)\mathrm{d}s]^{1/(1-m)}+\int_{t_i}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau=C+\sum_{i=1}^{k}\prod_{j=1}^{i-1}(1+\beta_j)^m\{[C^{1-m}+(1-m)\int_{t_0}^{t_i}P(s)\mathrm{d}s]^{1/(1-m)}-[C^{1-m}+(1-m)\int_{t_0}^{t_{i-1}}P(s)\mathrm{d}s]^{1/(1-m)}\}+\sum_{i=1}^{k}\beta_i\prod_{j=1}^{i-1}(1+\beta_j)[C^{1-m}+(1-m)\int_{t_0}^{t_i}P(s)\mathrm{d}s]^{1/(1-m)}+\int_{t_i}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau=(1+\beta_k)\prod_{t_0<t_i<t_k}(1+\beta_i)[C^{1-m}+(1-m)\int_{t_0}^{t_k}P(s)\mathrm{d}s]^{1/(1-m)}+\int_{t_k}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau.$$

再类似处理 $t\in I_1$ 时的方法，有

$$V(t)\leqslant\{\prod_{i=1}^{k}(1+\beta_i)^{1-m}[C^{1-m}+(1-m)\int_{t_0}^{t_k}P(s)\mathrm{d}s]+(1-m)\int_{t_k}^{t}P(\tau)\mathrm{d}\tau\}^{1/(1-m)}\leqslant\prod_{t_0<t_i<t}(1+\beta_i)[C^{1-m}+(1-m)\int_{t_0}^{t}P(s)\mathrm{d}s]^{1/(1-m)},t_k<t<t_{k+1}. \tag{2.10}$$

综合(2.6)式，(2.8)式，(2.9)式和(2.10)式可知，当 $0<m<1$ 时，估计(2.1)式成立.

再证明 $m>1$ 的情况. 当 $t\in I_1$ 时与 $0<m<1$ 的情况类似，由(1.1)式可以推出(2.5)式. 两边关于 t 积分，对于满足 $\int_{t_0}^{t}P(\tau)\mathrm{d}\tau<C^{1-m}/(m-1)$ 的任意 $t\in I_1$ 有

$$V(t)\leqslant C[1-(m-1)C^{m-1}\int_{t_0}^{t}P(\tau)\mathrm{d}\tau]^{-1/(m-1)}. \tag{2.11}$$

假设对满足

$$\int_{t_0}^{t}P(\tau)\mathrm{d}(\tau)<\frac{C^{1-m}}{(m-1)\prod_{t_0<t_i<t}(1+\beta_i)^{m-1}}$$

的任意 $t_0<t<t_k$ 有 $V(t)\leqslant C\prod_{t_0<t_i<t}(1+\beta_i)[1-(m-1)(C\prod_{t_0<t_i<t}(1+\beta_i))^{m-1}\int_{t_0}^{t}P(\tau)\mathrm{d}\tau]^{-1/(m-1)}$. 当 $t\in$

I_{k+1}，且满足$\int_{t_0}^{t}P(\tau)\mathrm{d}(\tau)<\frac{C^{1-m}}{(m-1)\prod_{t_0<t_i<t}(1+\beta_i)^{m-1}}$时，与$0<m<1$的情况类似，由(1.1)式推出

$$V(t)\leqslant C+\int_{t_0}^{t_k}P(\tau)\{C\prod_{t_0<t_i<\tau}(1+\beta_i)[1-(m-1)(C\prod_{t_0<t_i<\tau}(1+\beta_i))^{m-1}\times\int_{t_0}^{\tau}P(s)\mathrm{d}s]^{-1/(m-1)}\}^{m}\mathrm{d}\tau+\sum_{i=1}^{k}\beta_i C\prod_{t_0<t_j<t_i}(1+\beta_j)[1-(m-1)\times(C\prod_{t_0<t_j<t_i}(1+\beta_j))^{m-1}\int_{t_0}^{t_i}P(\tau)\mathrm{d}\tau]^{-1/(m-1)}+\int_{t_k}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau=C+\sum_{i=1}^{k}C\prod_{j=1}^{i-1}(1+\beta_j)\{[1-(m-1)(C\prod_{j=1}^{i-1}(1+\beta_j))^{m-1}\times\int_{t_0}^{t_i}P(s)\mathrm{d}s]^{-1/(m-1)}-[1-(m-1)\cdot(C\prod_{j=1}^{i-1}(1+\beta_j))^{m-1}\times\int_{t_0}^{t_{i-1}}P(s)\mathrm{d}s]^{-1/(m-1)}\}+\sum_{i=1}^{k}\beta_i C\prod_{j=1}^{i-1}(1+\beta_j)[1-(m-1)\times(C\prod_{j=1}^{i-1}(1+\beta_j))^{m-1}\int_{t_0}^{t_i}P(\tau)\mathrm{d}\tau]^{-1/(m-1)}+\int_{t_k}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau\leqslant C\prod_{j=1}^{k}(1+\beta_j)[1-(m-1)(C\prod_{j=1}^{k}(1+\beta_j))^{m-1}\int_{t_0}^{t_k}P(s)\mathrm{d}s]^{-1/(m-1)}+\int_{t_k}^{t}P(\tau)V^m(\tau)\mathrm{d}\tau.$$

用$C\prod_{j=1}^{k}(1+\beta_j)[1-(m-1)(C\prod_{j=1}^{k}(1+\beta_j))^{m-1}\int_{t_0}^{t_k}P(s)\mathrm{d}s]^{-1/(m-1)}$替换(2.11)式中的$C$得到

$$V(t)\leqslant\{C^{1-m}\prod_{j=1}^{k}(1+\beta_j)^{1-m}[1-(m-1)(C\prod_{j=1}^{k}(1+\beta_j))^{m-1}\int_{t_0}^{t_k}P(s)\mathrm{d}s]-(m-1)\int_{t_0}^{t}P(\tau)\mathrm{d}\tau\}^{-1/(m-1)}=[C^{1-m}\prod_{j=1}^{k}(1+\beta_j)^{1-m}-(m-1)\int_{t_0}^{t}P(s)\mathrm{d}s]^{-1/(m-1)}=C\prod_{j=1}^{k}(1+\beta_j)[1-(m-1)C^{m-1}\prod_{j=1}^{k}(1+\beta_j)^{m-1}\int_{t_0}^{t}P(s)\mathrm{d}s]^{-1/(m-1)}.\tag{2.12}$$

由(2.12)式不难证明估计式(2.3)．综上所述，定理1证明完毕．

参考文献：

[1] Agarwal R P, Deng S, Zhang W. Generalization of a retarded Gronwall-like inequality and its applications[J]. Appl Math Comput, 2005, 165: 599-612.

[2] Bellman R. The stability of solutions of linear differential equations[J]. Duke Math J, 1943, 10: 643-647.

[3] Borysenko S D. About asymptotical stability on linear approximation of the systems with impulse influence[J]. Ukra in Mat Zh, 1983, 35(2): 144-150.

[4] Borysenko S D, Ciarletta M, Iovane G. Integro-sum inequalities and motion stability of systems with impulse perturbations[J]. Nonlnear Anal, 2005, 62: 417-428.

[5] Gllo A, Piccirilo A M. About new analogies of Gronwall-Bellman-Bihari type inequalities for discontinuous functions and estimated solutions for impulsive differential systems[J]. Nonlnear Anal, 2007, 67: 1550-1559.

[6] Iovane G. Some new integral inequalities of Bellman-Bihari type with delay for discontinuous functions[J]. Nonlnear Anal, 2007, 66: 498-508.

[7] Wang W S. A generalized retarded Gronwall-like inequality in two variables and applications to BVP[J]. Appl Math Comput, 2007, 191(1): 144-154.

[8] 王五生．一个推广的具有双重和离散不等式及其应用[J]．四川大学学报：自然科学版，2007，44(4)：733-738.

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