Bifurcations of Travelling Wave Solutions for Third-order Nonlinear Schrödinger Equation*
三阶非线性 Schrödinger 方程行波解的分支

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Abstract: By using the bifurcation theory of dynamical systems to third-order nonlinear Schrödinger equation, the smooth solitary wave solutions, kink and anti-kink wave solutions and periodic wave solutions are obtained. Under different parametric conditions, various sufficient conditions to guarantee the existence of the above solutions are given. Some exact explicit parametric representations of the above waves are determined.
Key words: solitary wave solution, periodic wave solution, kink and anti-kink wave solution, third-order nonlinear Schrödinger equation
摘要:用动力系统分支理论研究了三阶非线性 Schrödinger 方程. 证明了该方程存在光滑孤立波解、扭结和反扭结波解和光滑周期波解. 在不同的参数条件下,给出了上述解存在的各类充分条件. 求出了该方程的显式精确行波解.
关键词:孤立波解 周期波解 扭结和反扭结波解 三阶非线性 Schrödinger 方程
中图法分类号:O175.2 **文献标识码**:A **文章编号**:1002-7378(2006)03-0148-05

1 Introduction

Karpman V. I et al[1] have recently considered the behavior of steady quasisolitons solutions in two cases for the third-order nonlinear Schrödinger (NLS) equation

$$i\partial_T\Psi + \frac{1}{2}\partial_X^2\Psi + |\Psi|^2\Psi + i\alpha_1|\Psi|^2\partial_X\Psi + i\alpha_2\Psi\partial_X|\Psi|^2 + i\alpha_3\partial_X^3\Psi = 0, \quad (1)$$

where α_1, α_2 and α_3 are real coefficients. Third-order nonlinear Schrödinger equation has different kinds of soliton and quasisoliton solutions[2].

In this paper, we study travelling wave solutions in the parameter space of this system.

Let $\Psi = \phi(\xi)e^{i(\lambda x - \omega t)}$, $\xi = x - ct$,

where c is wave speed, λ,ω is real constant. Then the Equation (1) becomes

$$-i(c\phi' + i\omega\phi) + \frac{1}{2}(\phi'' + 2\lambda\phi' i - \lambda^2\phi) + \phi^3 + i\alpha_1(\phi^2\phi' + \lambda\phi^3 i) + i2\alpha_2\phi^2\phi' + i\alpha_3(\phi''' + 3\lambda\phi'' i - 3\lambda^2\phi' - \lambda^3\phi) = 0, \quad (2)$$

where "prime" is the derivative with respect to ξ. Setting real part and imaginary part as 0, respectively, we have

$$(\frac{1}{2} - 3\alpha_3)\phi'' + (\omega - \frac{1}{2}\lambda^2 + \alpha_3\lambda^3)\phi + (1 - \alpha_1\lambda)\phi^3 = 0 \quad (3)$$

and

$$(\lambda - c - 3\alpha_3\lambda^2)\phi' + (\alpha_1 + 2\alpha_2)\phi^2\phi' + \alpha_3\phi''' = 0. \quad (4)$$

Integrating Equation (4) once and setting integration constants as 0, we have

$$\alpha_3\phi'' + (\lambda - c - 3\alpha_3\lambda^3)\phi + \frac{\alpha_1 + 2\alpha_2}{3}\phi^3 = 0. \quad (5)$$

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For

$$\frac{1}{2} - 3\alpha_3 = \alpha_3, \omega - \frac{1}{2}\lambda^2 + \alpha_3\lambda^3 = \lambda - c - 3\alpha_3,$$

$$1 - \alpha_1\lambda = \frac{\alpha_1 + 2\alpha_2}{3},$$

therefore

$$\alpha_3 = \frac{1}{8}, \omega - \frac{1}{2}\lambda^2 + \frac{1}{8}\lambda^3 = \lambda - c - \frac{3}{8}, 1 - \alpha_1\lambda = \frac{\alpha_1 + 2\alpha_2}{3},$$

we have the same Equations (3) and (5). Let

$$p = 8(\lambda - c - \frac{3}{8}), q = \frac{8(\alpha_1 + 2\alpha_2)}{3},$$

then Equation (5) becomes Equation (6)

$$\phi'' + p\phi + q\phi^3 = 0. \quad (6)$$

Obvious Equation (6) is equivalent to the Liénard system

$$\frac{d\phi}{d\xi} = y, \frac{dy}{d\xi} = - p\phi - q\phi^3, \quad (7)$$

with the first integral

$$H(\phi, y) = \frac{1}{2}y^2 + \frac{p}{2}\phi^2 + \frac{q}{4}\phi^4 = h. \quad (8)$$

We investigate the bifurcations of phase portraits of Equation (7) in the phase plane (ϕ, y) as the parameters (p, q) are changed. To investigate all possible bifurcations of solitary waves, kink and anti-kink waves and periodic waves of Equation (1), we need to find all periodic annuli and homoclinic orbits and heteroclinic orbits of Equation (7), which depend on the system parameters. The bifurcation theory of dynamical systems[3,4] plays an important role in our study.

2 Bifurcations of phase portraits of Equation (7)

For $pq < 0$, there exist three equilibrium points of Equation (7) at $\Phi(0,0)$, $A_{\pm}(\pm\sqrt{-\frac{p}{q}}, 0)$; for $pq > 0$, there exist one equilibrium point of Equation (7) at $\Phi(0,0)$; for $p = 0, q \neq 0$, there exist one equilibrium point of Equation (7) at $\Phi(0,0)$; $p \neq 0$, $q = 0$, there exist one equilibrium point of Equation (7) at $\Phi(0,0)$.

Let $M(\phi_i, 0)$ be the coefficient matrix of the linearized system of Equation (7) at an equilibrium point $(\phi_i, 0)$. Then we have

$$J(\phi_i, 0) = \det(M(\phi_i, 0)) = p + 3q\phi_i^2,$$

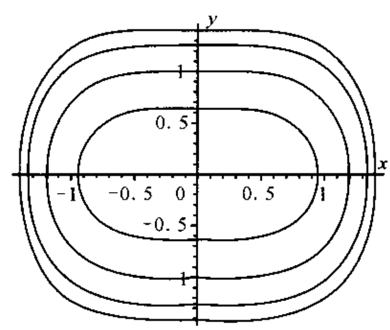
$$J(0,0) = p, \mathrm{Trace}(M(\phi_i, 0)) = 0.$$

By the theory of planar dynamical systems, we know that, if $J(0,0) = p > 0$ (or $J(0,0) = p < 0$), then equilibrium point $\Phi(0,0)$ is a center (or a saddle point); if $J(\pm\sqrt{-\frac{p}{q}}, 0) = p < 0$ (or $J(\pm\sqrt{-\frac{p}{q}}, 0) = p > 0$), then equilibrium point $A_{\pm}(\pm\sqrt{-\frac{p}{q}}, 0)$ is a center (or a saddle point); if $J(0,0) = p = 0, q \neq 0$, then equilibrium point $\Phi(0,0)$ is a high order equilibrium point. For $H(\phi, y)$ defined by Equation (8), we have

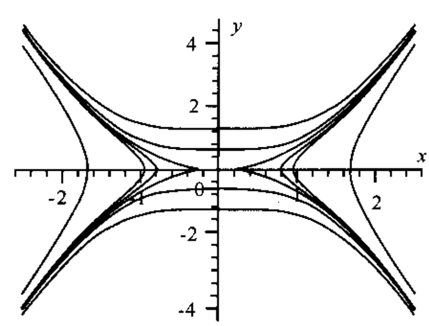
$$h_1 = H(0,0) = 0, h_2 = H(\pm\sqrt{-\frac{p}{q}}, 0) = \frac{p(p-2)}{4q}.$$

For a fixed h, the level curve $H(\phi, y) = h$ defined by Equation (8) determines a set of invariant curves of Equation (7). As h is varied, it defines different families of orbits of Equation (7) with different dynamical behaviors.

From the above analysis we obtain the different phase portraits of Equation (7) shown in Figures 1 and 2 and 3.

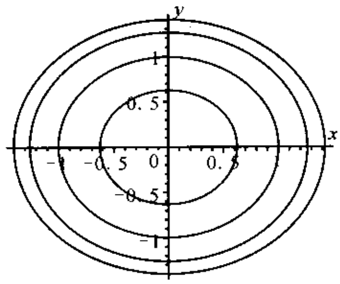


(a) $p = 0, q > 0$.

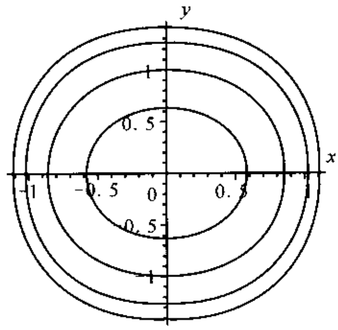


(b) $p = 0, q < 0$.

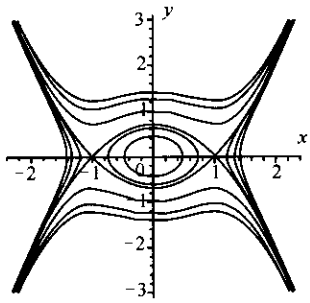
Fig. 1 Phase portraits of Equation (7) when $p = 0$



(a) $p > 0, q = 0$.

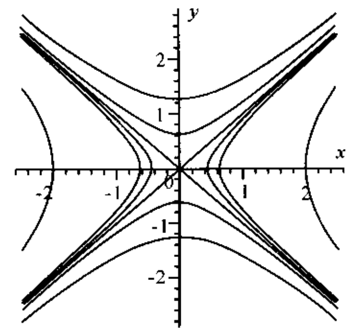


(b) $p > 0, q > 0$.

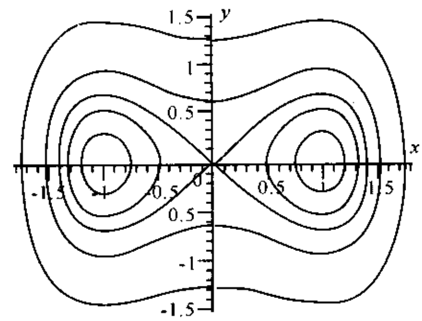


(c) $p > 0, q < 0$.

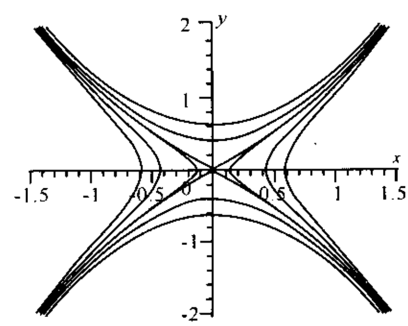
Fig. 2 Phase portraits of Equation (7) when $p > 0$



(a) $p < 0, q = 0$.



(b) $p < 0, q > 0$.



(c) $p < 0, q < 0$.

Fig. 3 Phase portraits of Equation (7) when $p < 0$

3 Exact explicit travelling wave solutions of Equation (1)

For $p = 0, q < 0$, Equations (7) and (8) become

$$\frac{d\phi}{d\xi} = y, \frac{dy}{d\xi} = -q\phi^3 \tag{12}$$

and

$$H(\phi, y) = \frac{1}{2}y^2 + \frac{q}{4}\phi^4 = h. \tag{13}$$

By using Systems (12) and (13) and the first equation of System (12), we can obtain some exact explicit parametric representations for the travelling wave solutions of Equation (1). For System (12), corresponding to $H(\phi, y) = h, H(0,0) = 0$ defined by System (13), we obtain the parametric representations of the arch orbit for the System (12) (Fig. 1b))

$$y^2 = -\frac{q}{2}\phi^4. \tag{14}$$

From Equation (14), we obtain with parametric representations for the exact explicit travelling wave solutions of Equation (1)

$$\phi = \pm\sqrt{\frac{-2}{q}}\frac{1}{\xi}. \tag{15}$$

It follows that Equation(1) has two exact explicit travelling wave solutions with the parametric representations

$$\phi(x - ct) = \pm\sqrt{\frac{-2}{q}}\frac{1}{x - ct}. \tag{16}$$

For $q = 0, p < 0$, Equations (7) and (8) become

$$\frac{d\phi}{d\xi} = y, \frac{dy}{d\xi} = -p\phi \tag{17}$$

and

$$H(\phi,y)=\frac{1}{2}y^2+\frac{p}{2}\phi^2=h. \tag{18}$$

By using Systems (18) and the first equation of System (17), we can obtain some exact explicit parametric representations for the travelling wave solutions of Equation (1). For System (17), corresponding to $H(\phi,y)=h, h\in(-\infty,+\infty)$ defined by System (18), we obtain the parametric representations of the arch orbit for the system (17) (Fig. 3a)

$$y^2=h-p\phi^2. \tag{19}$$

Thus, from Equation (14), we obtain the parametric representations of the arch orbit as follows:

$$\phi=\frac{1}{2}\left(e^{\pm\sqrt{-p}\xi}+\frac{h}{p}e^{\mp\sqrt{-p}\xi}\right). \tag{20}$$

It follows that Equation (1) has two exact explicit travelling wave solutions with the parametric representations

$$\phi(x-ct)=\frac{1}{2}\left(e^{\pm\sqrt{-p}(x-ct)}+\frac{h}{p}e^{\mp\sqrt{-p}(x-ct)}\right). \tag{21}$$

For $p>0, q<0$, corresponding to $H(\phi,y)=h_2$ defined by Equation (8), the Equation (7) has two heteroclinic orbits connecting the saddle points A_+ and A_-. Two orbits have the same algebraic equation as follows. (Fig. 2c)

$$y^2=-\frac{p^2}{2q}-p\phi^2-\frac{q}{2}\phi^4=-\frac{q}{2}\left(\phi^2+\frac{p}{q}\right)^2. \tag{22}$$

From Equation (22), we obtain the parametric representations of the arch orbit as follows:

$$\phi=\pm\frac{p}{2}\tanh\sqrt{\frac{p}{2}}\xi. \tag{23}$$

It follows that Equation (1) has two exact explicit travelling wave solutions with the parametric representations

$$\phi(x-ct)=\pm\frac{p}{2}\tanh\sqrt{\frac{p}{2}}(x-ct). \tag{24}$$

For $p<0, q>0$, corresponding to $H(\phi,y)=h_1$ defined by Equation (8), the Equation (7) has two homoclinic orbits connecting the saddle points $\Phi(0,0)$. Two orbits have the same algebraic equation as follows. (Fig. 3b)

$$y^2=-p\phi^2-\frac{q}{2}\phi^4=\frac{q}{2}\phi^2\left(-\phi^2+\frac{-2p}{q}\right). \tag{25}$$

Thus, from Equation (25), we obtain the parametric representations of the arch orbit as follows:

$$\phi=\sqrt{\frac{q}{-2p}}\operatorname{sech}\sqrt{\frac{-p}{2q}}\xi. \tag{26}$$

It follows that Equation (1) has one exact explicit travelling wave solution with the parametric representations

$$\phi(x-ct)=\sqrt{\frac{q}{-2p}}\operatorname{sech}\sqrt{\frac{-p}{2q}}(x-ct). \tag{27}$$

4 The existence of smooth travelling wave solutions of Equation (1)

In this section, we use the results of section 2 to discuss the existence of smooth solitary wave and periodic wave solutions. We first consider the existence of smooth periodic wave solutions and solitary wave solution and kink and anti-kink wave solution.

Theorem 4.1 (i) Suppose $p=0, q>0$, corresponding to a branch of the curves $H(\phi,y)=h$, $h\in(0,+\infty)$, defined by Equation (8), the Equation (1) has a smooth family of periodic wave solutions (Fig. 1a);

(ii) Suppose $p>0, q=0$, corresponding to a branch of the curves $H(\phi,y)=h, h\in(0,+\infty)$, defined by Equation (8), the Equation (1) has a smooth family of periodic wave solutions (Fig. 2a);

(iii) Suppose $p>0, q>0$, corresponding to a branch of the curves $H(\phi,y)=h, h\in(0,+\infty)$, defined by Equation (8), the Equation (1) has a smooth family of periodic wave solutions (Fig. 2b);

(iv) Suppose $p>0, q<0$, corresponding to a branch of the curves $H(\phi,y)=h, h\in(h_1,h_2)$, defined by Equation (8), the Equation (1) has a smooth family of periodic wave solutions (Fig. 2c);

(v) Suppose $p<0, q>0$, corresponding to a branch of the curves $H(\phi,y)=h, h\in(h_1,+\infty)$, or $h\in(h_2,h_1)$ defined by Equation (8), the Equation (1) has three smooth families of periodic wave solutions (Fig. 3b).

Theorem 4.2 (i) Suppose $p<0, q>0$, then, corresponding to a branch of the curves $H(\phi, y) = h_1$ defined by Equation (8), the Equation (1) has a pair smooth solitary wave solutions with peak type and valley type, respectively (Fig. 3b);

(ii) Suppose $p > 0, q < 0$, then, corresponding to a branch of the curves $H(\phi, y) = h_2$ defined by Equation (8), the Equation (1) has a smooth kink and anti-kink wave solutions (Fig. 2c).

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美国天文学家首次发现宇宙暗物质存在的直接证据

暗物质是指宇宙中存在的一种不明性质的物质粒子，它的电磁放射和折射非常微弱，所以不能被直接探测到。但是，我们肉眼能见的物质如星系、恒星、甚至各种生物，所占质量只是宇宙中很小一部分，暗物质的质量是可见的普通物质质量的5倍以上。暗物质不能被“看”，但可以通过引力异常测量到它存在的痕迹。天文学家尽管知道存在暗物质，但多年来一直没有发现它存在的直接证据。

最近美国亚利桑那大学的天文学家小组，用“钱德拉”X射线天文望远镜，观测距太阳系1亿光年处船底座两个星系团的碰撞、融合，发现了暗物质确实存在。

美国天文学家介绍说，这两个星系团相向高速运动，碰撞到一起时相对速度达到近每小时2000万公里。因此，构成星系团物质主体的炽热气体相互排斥，形成了外形像子弹头一般的气团，这个气团的主要成分是温度达数百万摄氏度的等离子氢、氦等。

“钱德拉”在X射线波段的观测表明，两个星系团在碰撞、融合时，主要物质都集中在子弹状的气团中。不过，当天文学家用美宇航局“哈勃”太空望远镜、欧洲南方天文台的“麦哲伦”天文望远镜从可见光波段观测这一区域时，却发现最大的质量并不在可见的气团中，而是分布在周围更广大的区域中。美国天文学家认为，以可见光波段观测这个区域，可以发现明显的“引力透镜”现象，也就是更遥远地方恒星发出的光在“路过”这个区域时，被大质量物质吸引而发生了扭曲。这个现象说明两个星系团周围存在大量的暗物质。暗物质粒子不与可见物质粒子发生作用，彼此之间也没有作用。因此，两个星系团中的可见物质碰撞、融合而形成子弹状气团，而暗物质依旧分布在周围。也只有在大质量星系团碰撞这样极其罕见的现象中，暗物质因为与可见物质明显不同，才能“偶露峥嵘”。

据《科学网》