组合数丢番图方程 $\binom{x}{2}=\binom{y}{4}$ 的正解

The Solution of Binomial Diophantine Equation $\binom{x}{2}=\binom{y}{4}$

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摘要:运用递推序列法,给出组合数丢番图方程 $\binom{x}{2}=\binom{y}{4}$ 的一个初等解法.
关键词:丢番图方程 递推序列 初等解法
中图法分类号:O156.7
Abstract:An elementary proof of the Diophantine equation $\binom{x}{2}=\binom{y}{4}$ is given by using recursion sequence method.
Key words:Diophantine equation,recusion sequence method,elementary proof

易知组合数丢番图方程

$$\binom{x}{2}=\binom{y}{4} \tag{1}$$

有解 $\binom{21}{2}=\binom{10}{4}=210$,以及平凡解 $\binom{2}{2}=\binom{4}{4}=1$ 和 $\binom{6}{2}=\binom{6}{4}=15$,

R. K. Guy[1]要求证明这是仅有的解.令 $u=2x-1$,$v=2y-3$.则易知方程(1)可化成以下方程

$$48u^2=v^4-10v^2+57, \tag{2}$$

它代表一个椭圆曲线.在有理数域上有 rank=2,1995 年 B. M. M. Weger[2]已经用 Baker 有效方法证明了:

定理 1 方程(2)仅有的正整数解是 $(u,v)=(1,1),(1,3),(3,5),(11,9)$和$(41,17)$.

由定理 1 可以获得

定理 2 设 $x>1$,$y>3$ 是正整数,则仅有的正整数解是$(x,y)=(2,4),(6,6),(21,10)$.

证明需要大量的计算.用 Baker 有效方法给出了方程(1)的解满足 $y<10^{2\times10^{17}}$,$x<10^{4\times10^{17}}$,然后用计算机并结合丢番图逼近技巧给出满足方程(1)的 x,y.

本文给出定理 2 从而也是定理 1 的初等解法的证明.

定理 2 的证明

方程(2)可化成以下方程

$$((v^2-5)/4)^2-3u^2=-2, \tag{3}$$

这是一个 Pell 方程,熟知(3)式的解由下列式子给出[3]:

$$\frac{v^2-5}{4}+u\sqrt{3}=(1+\sqrt{3})(2+\sqrt{3})^n, \text{(对某些整数 } n) \tag{4}$$

今证明(4)式仅当 $n=-1,0,1,2,3$ 时有解.

令 $$x_n+y_n\sqrt{3}=(2+\sqrt{3})^n, \tag{5}$$

则(4)式给出

$$v^2=4x_n+12y_n+5. \tag{6}$$

容易验证下列关系式:

$$x_{m+n}=x_mx_n+3y_my_n,\ y_{m+n}=x_my_n+x_ny_m, \tag{7}$$

$$x_{-n}=x_n,\ y_{-n}=-y_n, \tag{8}$$

$$x_{2n}=x_n^2+3y_n^2=2x_n^2-1=6y_n^2+1,\ y_n=2x_ny_n, \tag{9}$$

$$x_{n+2kr}\equiv(-1)^kx_n \pmod{x_r},\ y_{n+2kr}\equiv(-1)^ky_n \pmod{x_r}, \tag{10}$$

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$x_{n+2kr} \equiv x_n (\text{mod } y_r), y_{n+2kr} \equiv y_n (\text{mod } y_r)$, (11)

$x_{n+2} = 4x_{n+1} - x_n, x_0 = 1, x_1 = 2,$ (12)

$y_{n+2} = 4y_{n+1} - y_n, y_0 = 0, y_1 = 1,$ (13)

设 $4x_n + 12y_n = u_n$，则 $u_{n+2} = 4 \times u_{n+1} - u_n, u_0 = 4, u_1 = 20$.

熟知，对 $n = 0,1,2,3,\cdots$，$u_n (\text{mod } p)$ 将形成1个周期序列：$u_0, u_1, u_2, \cdots, u_{k-1}; u_0, u_1, \cdots, (\text{mod } p)$，周期为 k.

取 $p = 7,13,31,37,61,73,79,97,181,193,223,337,397,421,661,673,757,1201,1321,1567,2521,3697,6337,489061,549649,1166617$.

我们计算 $(u_n + 5)(\text{mod } p)$，及其周期 k，并计算当 $0 \leqslant n < k$，且其相应的 Jacobi 符号 $((u_n + 5)/p) = -1$ 的那些 n 值.

1：$n \equiv 5,6 (\text{mod } 8)$,

$u_n \equiv 6,6 (\text{mod } 7)$.

2：$n \equiv 7,8,9,10 (\text{mod } 12)$,

$u_n \equiv 11,7,7,11 (\text{mod } 13)$.

3：$n \equiv 4,9,10,11,18,20,23,24,26,27 (\text{mod } 32)$,

$u_n \equiv 11,13,24,11,22,30,23,23,17,30 (\text{mod } 31)$.

4：$n \equiv 4,5,6,11,12,13,25,26,27,28,23,30 (\text{mod } 36)$,

$u_n \equiv 29,39,6,6,39,29,35,15,15,35,8,8 (\text{mod } 37)$.

5：$n \equiv 4,6,11,12,17,18,23,25,39,40,49,50 (\text{mod } 60)$,

$u_n \equiv 28,7,30,37,37,30,7,28,63,59,59,63 (\text{mod } 61)$.

6：$n \equiv 24,29 (\text{mod } 36)$,

$u_n \equiv 60,60 (\text{mod } 73)$.

7：$n \equiv 9,12,19,25,33,47,48,51,56,57,60,63,67,71,72,73,76 (\text{mod } 80)$,

$u_n \equiv 43,15,35,71,75,47,6,63,39,70,54,39,74,6,47,14,37 (\text{mod } 79)$.

8：$n \equiv 9 (\text{mod } 16)$,

$u_n \equiv 82 (\text{mod } 97)$.

9：$n \equiv 4,5,12,14,15,16,17 (\text{mod } 20)$,

$u_n \equiv 160,160,110,31,31,83,110 (\text{mod } 181)$.

10：$n \equiv 16,17,18 (\text{mod } 24)$,

$u_n \equiv 103,102,102 (\text{mod } 193)$.

11：$n \equiv 7,8,15,16,17,19,28,31,34,39,40,44,47,48,51,60,63,64,65,67,71,72,80,81,83,92,95,96,97,103,104,113,115,124,129,136,144,159,163,172,176,191,194,199,209,211,220 (\text{mod } 224)$,

$u_n \equiv 24,35,117,195,207,45,118,186,161,158,157,13,189,67,48,48,67,189,10,13,157,158,186,150,118,45,195,117,40,35,24,208,167,151,26,70,95,44,185,185,44,95,52,70,193,151,167 (\text{mod } 223)$.

12：$n \equiv 7,12,15,19,20,31,32,33,36,42,49,50,52 (\text{mod } 56)$,

$u_n \equiv 174,291,291,71,174,58,293,93,276,22,69,93,58 (\text{mod } 337)$.

13：$n \equiv 14,15,16,36,37,39,52,74,88,107,109,123,144,145,158,160,181,182,183,216,217,218,219,232,251,252,254,255,267,290,291,302,303,323,326,338,339,359,360,361,374,375,376 (\text{mod } 396)$,

$u_n \equiv 302,380,17,166,356,259,214,198,242,263,242,198,345,214,259,356,17,380,302,134,194,235,339,185,62,45,377,199,245,203,18,18,203,161,245,199,377,241,109,185,339,235,194 (\text{mod } 397)$.

14：$n \equiv 27,28,86,107,121,122,123,168,179,242,268,303,326,359,361,387 (\text{mod } 420)$,

$u_n \equiv 206,209,224,306,111,402,224,168,156,200,73,298,298,264,73,200 (\text{mod } 421)$.

15：$n \equiv 47 (\text{mod } 60)$,

$u_n \equiv 353 (\text{mod } 661)$.

16：$n \equiv 48,99,242,279 (\text{mod } 336)$,

$u_n \equiv 316,211,558,328 (\text{mod } 673)$.

17：$n \equiv 27,72 (\text{mod } 84)$,

$u_n \equiv 242,486 (\text{mod } 757)$.

18：$n \equiv 26,98,111 (\text{mod } 120)$,

$u_n \equiv 974,314,941 (\text{mod } 1201)$.

19：$n \equiv 22,43,61,79,99,102,122,142,143,180,199,200,201,203 (\text{mod } 220)$,

$u_n \equiv 584,342,466,252,217,944,560,1133,664,208,283,862,513,234 (\text{mod } 1321)$.

20：$n \equiv 35,167,168,192,208 (\text{mod } 224)$,

$u_n \equiv 325,1155,1155,903,1206 (\text{mod } 1567)$.

21：$n \equiv 7 (\text{mod } 28)$,

$u_n \equiv 2164 (\text{mod } 2521)$.

22:$n\equiv 27,28,29,31,83,86,196,199,223,227,283 \pmod{308}$,

$u_n \equiv 1544,3088,3404,1688,2126,2750,874,724,2214,3368,733 \pmod{3697}$.

23:$n\equiv 33,194,226,323 \pmod{352}$,

$u_n\equiv 5004,4733,5045,4558 \pmod{6337}$.

24:$n\equiv 38 \pmod{44}$,

$u_n\equiv 485110 \pmod{489061}$.

25:$n\equiv 192 \pmod{264}$,

$u_n\equiv 323782 \pmod{549649}$.

26:$n\equiv 266 \pmod{396}$,

$u_n\equiv 904805 \pmod{1166617}$.

综上即知,如果(4)式有解,则必有 $n\equiv 0,1,2,3,55439,55440,110879 \pmod{110880}$ 或 $n\equiv 0,1,2,3,-1 \pmod{55440}$;

27:设,$n=2+2^s a(2h+1)$,$s\geqslant 2$,

$v^2=4x_n+12y_n+5\equiv(-1)^{2h+1}x_2+(-1)^{2h+1}y_2+5=-4\times 26-12\times 15+5 \pmod{x_{a\cdot 2^s}}$,

即,$v^2\equiv -71 \pmod{x_{a\cdot 2^s}}$,

$$1=\left(\frac{v^2}{x_{a2^s}}\right)=\left(\frac{-71}{x_{a2^s}}\right)=\left(\frac{x_{a2^s}}{71}\right).$$

取 $a=1,3,5,9,15$. 从表 1 可知上式不成立.

28:设 $n=a2^{s+1}(4h\pm 1)+3\neq 3$,$n=2k+3$,$k=a\cdot 2^s(4h\pm 1)$,

$v^2=4(x_{2k}x_3+3y_{2k}y_3)+12(x_{2k}y_3+x_3y_{2k})+5=4(26x_{2k}+45y_{2k})+12(15x_{2k}+26y_{2k})+5=284x_{2k}+492y_{2k}+5=284(x_k^2+3y_k^2)+2\times 492x_ky_k+5(x_k^2-3y_k^2)=289x_k^2+2\times 492x_ky_k+837y_k^2$,

$v^2\equiv 289x_{a\cdot 2^s}^2\pm 2\times 492x_{a\cdot 2^s}y_{a\cdot 2^s}+837y_{a\cdot 2^s}^2 \pmod{x_{a\cdot 2^s}^2+3y_{a\cdot 2^s}^2}$,

$v^2\equiv 10x_{a\cdot 2^s}^2\pm 2\times 492x_{a\cdot 2^s}y_{a\cdot 2^s}\equiv 2x_{a\cdot 2^s}(5x_{a\cdot 2^s}\pm 492y_{a\cdot 2^s}) \pmod{x_{a\cdot 2^s}^2+3_{a\cdot 2^s}^2}$,

$$1=\left(\frac{v^2}{x_{a\cdot 2^s}^2+3y_{a\cdot 2^s}^2}\right)=$$
$$\left(\frac{2x_{a\cdot 2^s}}{x_{a\cdot 2^s}^2+3y_{a\cdot 2^s}^2}\right)\left(\frac{5x_{a\cdot 2^s}\pm 492y_{a\cdot 2^s}}{x_{a\cdot 2^s}^2+3y_{a\cdot 2^s}^2}\right)=$$
$$\left(\frac{x_{a\cdot 2^s}}{x_{a\cdot 2^s}^2+3y_{a\cdot 2^s}^2}\right)\left(\frac{5x_{a\cdot 2^s}\pm 492y_{a\cdot 2^s}}{x_{a\cdot 2^s}^2+3y_{a\cdot 2^s}^2}\right)=$$
$$\left(\frac{x_{a\cdot 2^s}^2+3y_{a\cdot 2^s}^2}{x_{a\cdot 2^s}^2}\right)\left(\frac{x_{a\cdot 2^s}^2+3y_{a\cdot 2^s}^2}{5x_{a\cdot 2^s}\pm 492y_{a\cdot 2^s}}\right)\left(\frac{x_{a\cdot 2^s}}{3}\right)$$
$$\left(\frac{492^2+3\times 5^2}{5x_{a\cdot 2^s}\pm 492y_{a\cdot 2^s}}\right)=\left(\frac{242139}{5x_{a\cdot 2^s}\pm 492y_{a\cdot 2^s}}\right)=$$
$$\left(\frac{5x_{a\cdot 2^s}\pm 492y_{a\cdot 2^s}}{242139}\right)=\left(\frac{5x_{a\cdot 2^s}\pm 492y_{a\cdot 2^s}}{80713}\right)$$

当“+”号时取 $a=9,15,21,35,45,63,105,315$. 从表 2 可知上式不成立.

当“—”号时取 $a=7,9,15,21,35,45,63,105,315$. 从表 3 可知上式不成立.

表 1 Jacobi 符号

s	1	3	5	9	15	
0:	2	26*	7*	7*	2	−1
1:	7*	2	26*	26*	7*	−3
2:	26*	7*	2	2	26*	−1
3:	2	26*	7*	7*	2	−1
4:	7*	2	26*	26*	7*	−3
5:	26*	7*	2	2	26*	−1
6:	2	26*	7*	7*	2	−1

* 表示 Jacobi 符号 $(a/p)=-1$.

表 2 Jacobi 符号

s	9	15	21	35	45	63	105	315
0:	201733*	20167*	50365	40906	118819	145537	232585*	71551*
1:	182756*	78551*	181229	162404	60317	141041*	4271	73697*
2:	94208*	223442	179318	216005	16460	216164	225530*	162773
3:	128390*	216263*	176522	212288*	75893*	93263*	36002	220328*
…	…	…	…	…	…	…	…	…
524:	94208*	223442	179318	216005	16460	216164	225530*	162773
525:	128390*	216263*	176522	212288*	75893*	93263*	36002	220328*

* 表示 Jacobi 符号 $(a/p)=-1$,其中省略部分:对每一个 $s=6,\ldots,519$ 至少有一列带有 * 号.

表 3 Jacobi 符号

s	7	9	15	21	35	45	63	105	315
0:	45832*	16249	84889	48196	109717*	88360*	2275*	185194	195808*
1:	71048*	12077*	225509*	40136*	46202	61742*	89735	175919*	12998
2:	94649	52865*	54950*	221570*	165122*	191459*	134357*	20117	85049
3:	116300	103403*	112625*	239750	186497*	219503*	199445*	197522*	137057*
…	…	…	…	…	…	…	…	…	…
524:	94649	52865*	54950*	221570*	165122*	191459*	134357*	20117	85049
525:	116300	103403*	112625*	239750	186497*	219503*	199445*	197522*	137057*

* 表示 Jacobi 符号 $(a/p)=-1$.

29：设 $n = a2^{s+1}(4h \pm 1) + 1 \neq 1, n = 2k + 1$，$k = a \cdot 2^s(4h \pm 1)$，

$$v^2 = 4(x_{2k}x_1 + 3y_{2k}y_1) + 12(x_{2k}y_1 + x_1y_{2k}) + 5 = 4(2x_{2k} + 3y_{2k}) + 12(x_{2k} + 2y_{2k}) + 5 = 20x_{2k} + 36y_{2k} + 5 = 20(x_x^2 + 3y_k^2) + 2 \times 36x_ky_k + 5(x_k^2 - 3y_k^2) = 25x_k^2 + 2 \times 36x_ky_k + 45y_k^2,$$

$$v^2 \equiv 25x_{a\cdot 2^s}^2 \pm 2 \times 36x_{a\cdot 2^s}y_{a\cdot 2^s} + 45y_{a\cdot 2^s}^2 \equiv 10x_{a\cdot 2^s}^2 \pm 2 \times 36x_{a\cdot 2^s}y_{a\cdot 2^s} (\mathrm{mod}\ x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2),$$

$$v^2 \equiv 2x_{a\cdot 2^s}(5x_{a\cdot 2^s} \pm 36y_{a\cdot 2^s}) (\mathrm{mod}\ x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2),$$

$$1 = \left(\frac{v^2}{x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2}\right) = \left(\frac{2x_{a\cdot 2^s}(5x_{a\cdot 2^s} \pm 36y_{a\cdot 2^s})}{x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2}\right) = \left(\frac{5x_{a\cdot 2^s} \pm 36y_{a\cdot 2^s}}{x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2}\right) = \left(\frac{x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2}{5x_{a\cdot 2^s} \pm 36y_{a\cdot 2^s}}\right) = \left(\frac{(36^2 + 3.5^2)y_{a\cdot 2^s}^2}{5x_{a\cdot 2^s} \pm 36y_{a\cdot 2^s}}\right) = \left(\frac{5x_{a\cdot 2^s} \pm 36y_{a\cdot 2^s}}{1371}\right),$$

取 $a = 1,3,15,35,45$. 从表4可知上式不成立.

30：设 $n = a2^{s+1}(4h \pm 1) - 1 \neq 1, n = 2k - 1$，$k = a \cdot 2^s(4h \pm 1)$，

表4　Jacobi 符号

s	1	3	15	35	45	1	3	15	35	45
0:	46*	670*	631	1033	469	1345*	961	499	772*	1147
1:	539	236	1148	245	941*	902*	935*	587*	125	86
2:	920*	749*	347*	746*	1052	50*	386	1190	575	395
3:	536*	503	149	398	1175*	1178	755	965*	92	1067*
…	…	…	…	…	…	…	…	…	…	…
20:	920*	749*	347*	746*	1052	50*	386	1190	575	395
21:	536*	503	149	398	1175*	1178	755	965*	92	1067*

* 表示 Jacobi 符号 $(a/p) = -1$.

$$v^2 = 4(x_{2k}x_{-1} + 3y_{2k}y_{-1}) + 12(x_{2k}y_{-1}) + 12(x_{2k}y_{-1} + x_{-1}y_{2k}) + 5 = 4(2x_{2k} - 3y_{2k}) + 12(-x_{2k} + 2y_{2k}) + 5 = -4x_{2k} + 12y_{2k} + 5 = -4(x_k^2 + 3y_k^2) + 2 \times 12x_ky_k + 5(x_k^2 - 3y_k^2) = x_k^2 + 2 \times 12x_ky_k - 27y_k^2,$$

$$v^2 \equiv x_{a\cdot 2^s}^2 \pm 2 \times 12x_{a\cdot 2^s}y_{a\cdot 2^s} - 27_{a\cdot 2^s}^2 \equiv 10x_{a\cdot 2^s}^2 \pm 2 \times 12x_{a\cdot 2^s}y_{a\cdot 2^s} (\mathrm{mod}\ x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2),$$

$$v^2 \equiv 2x_{a\cdot 2^s}(5x_{a\cdot 2^s} \pm 12y_{a\cdot 2^s}) (\mathrm{mod}\ x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2),$$

$$1 = \left(\frac{v^2}{x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2}\right) = \left(\frac{2x_{a\cdot 2^s}(5x_{a\cdot 2^s} \pm 12y_{a\cdot 2^s})}{x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2}\right) = \left(\frac{5x_{a\cdot 2^s} \pm 12y_{a\cdot 2^s}}{x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2}\right) = \left(\frac{x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2}{5x_{a\cdot 2^s} \pm 12y_{a\cdot 2^s}}\right) = \left(\frac{(12^2 + 3.5^2)y_{a\cdot 2^s}^2}{5x_{a\cdot 2^s} \pm 12y_{a\cdot 2^s}}\right) = \left(\frac{5x_{a\cdot 2^s} \pm 12y_{a\cdot 2^s}}{219}\right),$$

取 $a = 1,3,15,35,45$. 从表5可知上式不成立.

表5　Jacobi 符号

s	1	3	15	35	45	1	3	15	35	45
0:	22*	91	196	217	214*	217	169	55	22*	151*
1:	203*	146	146	86	5	86	5	5	203*	131
2:	8*	146	146	86	98*	86	68	68	8*	131
3:	146	68	68	122*	191	122*	146	146	146	38*
…	…	…	…	…	…	…	…	…	…	…
20:	8*	146	146	86	98*	86	68	68	8*	131
21:	146	68	68	122*	191	122*	146	146	146	38*

* 表示 Jacobi 符号 $(a/p) = -1$.

31：设 $n = a \cdot 2^{s+1}(4h \pm 1) \neq 0, n = 2k, k = a \cdot 2^s(4h \pm 1), s \geqslant 2$，

$$v^2 = 4(x_k^2 + 3y_k^2) + 2 \times 12x_ky_k + 5(x_k^2 - 3y_k^2) = 9x_k^2 + 2 \times 12x_ky_k - 3y_k^2,$$

$$v^2 \equiv 9x_{a\cdot 2^s}^2 \pm 2 \times 12x_{a\cdot 2^s}y_{a\cdot 2^s} - 3y_{a\cdot 2^s}^2 \equiv 10x_{a\cdot 2^s}^2 \pm 2 \times 12x_{a\cdot 2^s}y_{a\cdot 2^s} (\mathrm{mod}\ x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2),$$

$$v^2 \equiv 2x_{a\cdot 2^s}(5x_{a\cdot 2^s} \pm 12y_{a\cdot 2^s}) (\mathrm{mod}\ x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2),$$

$$1 = \left(\frac{v^2}{x_{a\cdot 2^s}^2 + 3y_{a\cdot 2^s}^2}\right) = \left(\frac{5x_{a\cdot 2^s} \pm 12y_{a\cdot 2^s}}{219}\right),$$

从表5可知上式不成立.

综上所述，如果 $n \neq -1,0,1,2,3$ 则(6)式无解，而当 $n = -1,0,1,2,3$ 时正好给出定理1的解.

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