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Structures of Grouplike Coalgebras*
群像余代数的结构研究

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Abstract:In this paper,we investigate the sructures of grouplike coalgebras $K[S]$ and $K[G]$, where S is a non-empty set and G is a monoid whose identity e is the only invertible element of G, and obtain the conclusion that the K - linear isomorphism $K[G\times G']\simeq K[G]\otimes K[G']$ defined by $f'(g,g')=g\otimes g'$, for all $g\in G,g'\in G'$ is an isomorphism of coalgebra.

Key words:grouplike coalgebra,tensor product,isomorphism of coalgebra

摘要:研究群像余代数 $K[S]$ 和 $K[G]$ 的结构,其中 S 是一个非空集合,G 是一个只有单位元和逆元的幺半群,得到结论:对任意 $g\in G,g'\in G'$,定义 $f'(g,g')=g\otimes g'$,则线性同构 $k[G\times G']\simeq k[G]\otimes k[G']$ 是余代数同构.

关键词:群像余代数　张量积　余代数同构

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0 Introduction

In this paper, K is a field,and it is well known that for any non-empty set S the K - vector space $C=K[S]$ (reference[1]), with basis S is a coalgbra whose comultiplication and counit are defined by

$$\Delta_C(s)=s\otimes s,\varepsilon_C(s)=1,\text{ for all } s\in S.$$

Let G be a monoid whose identity e is the only invertible element of G ,and all $g\in G$ have only finitely many factorizations $g=ab$ where $a,b\in G$. There may be other ways of putting a coalgebra structure on the K - module $M=K[G]$ (reference[1]). It is also a coalgebra whose comultiplication and counit are defined by

$$\Delta_M(g)=\sum_{ab=g}a\otimes b,\varepsilon_M(g)=\delta_{g,e},\text{ for all } g\in G.$$

Let $\{\alpha=\{\alpha_i\},i\in I\}$ be a basis of K -vector space V ,by putting a coalgebra structure on it, V forms a coalgebra and $\Delta(\alpha_i)=\alpha_i\otimes\alpha_i,\varepsilon(\alpha_i)=1$,for all $\alpha_i\in\alpha$. This coalgebra is called grouplike colalgebra. So $K[S]$ and $K[G]$ are grouplike colalgebras.

Let I be a K - subspace of $K[S]$,then I is a coideal of $K[S]$ if $\triangle_C(I)\subseteq I\otimes K[S]+K[S]\otimes I$ and $\varepsilon_C(I)=0$. Moreover,if C' is a K - subspace of $K[S]$ and $\triangle_C(C')\subseteq C'\otimes C'$,then C' is a subcoalgebra of $K[S]$. By using K -linear map,we can discuss the morphism of coalgebra. Let $(C,\Delta_C,\varepsilon_C)$, $(D,\Delta_D,\varepsilon_D)$ be two K -coalgebras. The K -linear map $f:C\to D$ is a morphism of coalgebra if $\Delta_D f=(f\otimes f)\Delta_C$ and $\varepsilon_D f=\varepsilon_C$ (reference[2]).

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1 The sructure of $K[S]$

In this section, we determine the coideals, subcoalgebras, quotients, Cartensian products[3] and tensor products [4,5] of grouplike coalgebra $K[S]$, where S is a non-empty set. Let S and T are non-empty sets, from the K-linear isomorphism $K[S\times T]\simeq K[S]\otimes K[T]$, we obtain an isomorphism of coalgebra.

Proposition 1.1[1,2] For any non-empty S and any $s,s'\in S, K(s-s')$ is a coideal of $K[S]$.

So C_1 is a coideal of $K[S]$.

Proposition 1.2[1,2] Every subcoalgebra of $K[S]$ has the form $K[S']$, where S' is a subset of S.

From this, it follows that for any subcoalgebra C' of $K[S]$ and any element $c_i'=\sum k_{ij}s_{ij}, k_{ij}\in K, s_{ij}\in S$, we have $\Delta_C(c_i')\subseteq C'\otimes C'$. Let $S'=\{s_{ij}\}$, $K[S']$ is a K-subspace of $K[S]$.

Proposition 1.3 Every quotient of $K[S]$ is also grouplike coalgebra.

Proof Let I be a coideal of $C=K[S]$, then C/I is a coalgebra over K defined by $\overline{\triangle_C}(\bar{s})=\overline{\triangle_C(s)}=\overline{s\otimes s}=\bar{s}\otimes\bar{s}, \bar{\varepsilon_C}(\bar{s})=\overline{\varepsilon_C(s)}=\bar{1}$, for any $\bar{s}\in\bar{S}=S+I$. We only need to prove that C/I is a coalgebra. For any $\bar{c}=C/I, \bar{c}=\sum k_i\bar{s_i}$, we can get

$$(\overline{\triangle_C}\otimes 1)\overline{\triangle_C}(\bar{c})=(\overline{\triangle_C}\otimes 1)\overline{\triangle_C}(\sum k_i\bar{s_i})=\sum k_i(\overline{\triangle_C}\otimes 1)\overline{\triangle_C}(\bar{s_i}))=$$
$$\sum k_i(\overline{\triangle_C}\otimes 1)(\bar{s_i}\otimes\bar{s_i})=\sum k_i(\overline{\triangle_C}(\bar{s_i})\otimes\bar{s_i})=$$
$$\sum k_i(\bar{s_i}\otimes\bar{s_i}\otimes\bar{s_i})=\sum k_i(\bar{s_i}\otimes\overline{\triangle_C}(\bar{s_i}))=$$
$$(1\otimes\overline{\triangle_C})(\sum k_i(\bar{s_i}\otimes\bar{s_i}))=(1\otimes\overline{\triangle_C})\overline{\triangle_C}\cdot(\sum k_i\bar{s_i})=(1\otimes\overline{\triangle_C})\overline{\triangle_C}(\bar{c}).$$

The coassociation is checked. The second condition from the definition of coalgebra is equivalent to that

$$(1\otimes\overline{\varepsilon_C})\overline{\triangle_C}(\bar{c})=(1\otimes\overline{\varepsilon_C})\overline{\triangle_C}(\sum k_i\bar{s_i})=\sum k_i\overline{\varepsilon_C}(\bar{s_i})\bar{s_i}=\sum k_i\bar{s_i}=\bar{c}.$$

Similarly, $(\overline{\varepsilon_C}\otimes 1)\overline{\triangle_C}(\bar{c})=\bar{c}$. So the proof is complete.

Lemma 1.1 For any non-empty sets S and T, $K[S\times T]$ is a grouplike coalgebra defined by $\Delta_{S\times T}((s,t))=(s,t)\otimes(s,t), \varepsilon_{S\times T}((s,t))=1$, for all $s\in S, t\in T$.

Proof For any $a\in K[S\times T], a=\sum k_{ij}(s_i,t_j), k_{ij}\in K, s_i\in S, t_j\in T$,

$$(1\otimes\Delta_{S\times T})\Delta_{S\times T}(a)=$$
$$(1\otimes\Delta_{S\times T})\Delta_{S\times T}(\sum k_{ij}(s_i,t_j))=$$
$$\sum k_{ij}(1\otimes\Delta_{S\times T})((s_i,t_j)\otimes(s_i,t_j))=$$
$$\sum k_{ij}((s_i,t_j)\otimes\Delta_{S\times T}((s_i,t_j)))=$$
$$\sum k_{ij}(s_i,t_j)\otimes(s_i,t_j)\otimes(s_i,t_j)=$$
$$\sum k_{ij}(\Delta_{S\times T}\otimes 1)((s_i,t_j)\otimes(s_i,t_j))=$$
$$(\Delta_{S\times T}\otimes 1)\Delta_{S\times T}(\sum k_{ij}(s_i,t_j))=$$
$$(\Delta_{S\times T}\otimes 1)\Delta_{S\times T}(a).$$

Showing that $\Delta_{S\times T}$ is coassociative. We also have

$$(1\otimes\varepsilon_{S\times T})\Delta_{S\times T}(a)=\sum k_{ij}(1\otimes\varepsilon_{S\times T})((s_i,t_j)\otimes(s_i,t_j))=\sum k_{ij}(s_i,t_j)=a.$$

Analogously, $(\varepsilon_{S\times T}\otimes 1)\Delta_{S\times T}(a)=a$. So $K[S\times T]$ is a grouplike coalgebra.

Lemma 1.2 For any non-empty sets S and T, $K[S]\otimes K[T]$ is a grouplike coalgebra defined by $\Delta_{S\otimes T}(s\otimes t)=s\otimes t\otimes s\otimes t, \varepsilon_{S\otimes T}=1$, for all $s\in S$, $t\in T$.

Proof For any $a'\in K[S]\otimes K[T], a'=(\sum k_i s_i)\otimes(\sum l_j t_j)=\sum k_i l_j(s_i\otimes t_j) k_i, l_j\in K, s_i\in S, t_j\in T$. From the definition of $\Delta_{S\otimes T}$ and $\varepsilon_{S\otimes T}$, it follows that

$$(1\otimes\Delta_{S\otimes T})\Delta_{S\otimes T}(a')=$$
$$(1\otimes\Delta_{S\otimes T})\Delta_{S\otimes T}(\sum k_i l_j(s_i\otimes t_j))=$$
$$\sum k_i l_j(1\otimes\Delta_{S\otimes T})(s_i\otimes t_j\otimes s_i\otimes t_j)=$$
$$\sum k_i l_j(s_i\otimes t_j)\otimes\Delta_{S\otimes T}(s_i\otimes t_j))=\sum k_i l_j(s_i\otimes t_j\otimes s_i\otimes t_j\otimes s_i\otimes t_j)=\sum k_i l_j(\Delta_{S\otimes T}\otimes 1)((s_i\otimes t_j)\otimes(s_i\otimes t_j))=(\Delta_{S\otimes T}\otimes 1)\Delta_{S\otimes T}(\sum k_i l_j(s_i\otimes t_j))=(\Delta_{S\otimes T}\otimes 1)\Delta_{S\otimes T}(a'),$$
$$(1\otimes\varepsilon_{S\otimes T})\Delta_{S\otimes T}(a')=\sum k_i l_j(1\otimes\varepsilon_{S\otimes T})((s_i\otimes t_j)\otimes(s_i\otimes t_j))=\sum k_i l_j(s_i\otimes t_j)=a'.$$

Similarly, $(\varepsilon_{S\otimes T}\otimes 1)\Delta_{S\otimes T}(a')=a'$. So $K[S]\otimes K[T]$ is a grouplike coalgebra.

The construction of coalgebra described above behaves well with respecting to morphism. From Lemma 1.1 and Lemma 1.2, it is no difficult to observe the following theorem.

Theorem 1.1 The K-linear isomorphism $K[S\times T]\simeq K[S]\otimes K[T]$ defined by $f((s,t))=s\otimes t$, for all $s\in S, t\in T$ is isomorphism of coalgebra.

Proof We show that f is a coalgebra map. For any $a \in K[S\times T], a=\sum k_{ij}(s_i,t_j), k_{ij}\in K, s_i\in S, t_j\in T$,

$$\varepsilon_{S\otimes T}f(a)=\varepsilon_{S\otimes T}f(\sum k_{ij}(s_i,t_j))= \sum k_{ij}\varepsilon_{S\otimes T}f((s_i,t_j))=\sum k_{ij}\varepsilon_{S\otimes T}(s_i\otimes t_j)= \sum k_{ij}=\sum k_{ij}\varepsilon_{S\times T}((s_i,t_j))=\varepsilon_{S\times T}(\sum k_{ij}(s_i,t_j))= \varepsilon_{S\times T}(a).$$

Moreover,

$$(f\otimes f)\Delta_{S\times T}(a)=(f\otimes f)\Delta_{S\times T}(\sum k_{ij}(s_i, t_j))=\sum k_{ij}(f\otimes f)((s_i,t_j)\otimes(s_i,t_j))= \sum k_{ij}((s_i\otimes t_j)\otimes(s_i\otimes t_j))= \sum k_{ij}\Delta_{S\otimes T}f((s_i,t_j))=\Delta_{S\otimes T}f(\sum k_{ij}(s_i,t_j))= \Delta_{S\otimes T}f(a).$$

It is easy to check that f is bijective.

2 The sructure of $K[G]$

Let G is a monoid whose identity is the only invertible element of G, the K-space $K[G]$ is grouplike coalgebra with Δ_M and ε_M. Similarly, we determine the Cartensian product and tensor product of $K[G]$.

From theorem 1.1, it is easy to show that there also has isomorphism of coalgebra $K[G\times G']\simeq K[G]\otimes K[G']$, where G and G' are monoids whose identity is the only invertible element.

Lemma 2.1 $K[G\times G']$ is grouplike coalgebra defined by

$$\Delta_{G\times G'}((g,g'))=\sum_{\substack{g_1g_2=g\\ g'_1g'_2=g'}}((g_1,g_1')\otimes(g_2, g_2')), \varepsilon_{G\times G'}((g,g'))=\delta_{g,e_G}\delta_{g',e_{G'}}.$$

Proof For any $k(g,g')\in K[G\times G'], k\in K, g\in G, g'\in G'$,

$$(1\otimes\Delta_{G\times G'})\Delta_{G\times G'}(k(g,g'))=k(1\otimes \Delta_{G\times G'})\Delta_{G\times G'}((g,g'))=k(1\otimes\Delta_{G\times G'})(\sum_{\substack{g_1g_2=g\\ g'_1g'_2=g'}}(g_1, g'_1)\otimes(g_2,g'_2))=k\sum_{\substack{g_1g_2=g\\ g'_1g'_2=g'}}((g_1,g'_1)\otimes \Delta_{G\times G'}((g_2,g'_2)))=k\sum_{\substack{g_1g_2=g\\ g'_1g'_2=g'\\ g_3g_4=g_2\\ g'_3g'_4=g'_2}}((g_1,g'_1)\otimes (g_3,g'_3)\otimes(g_4,g'_4))=k\sum_{\substack{g_1g_3=g_5\\ g'_1g'_3=g'_5\\ g_5g_4=g\\ g'_5g'_4=g'}}(\Delta_{G\times G'}((g_5, g_5'))\otimes(g_4,g_4'))=k(\Delta_{G\times G'}\otimes 1)(\sum_{\substack{g_5g_4=g\\ g'_5g'_4=g'}}(g_5, g'_5)\otimes(g_4,g'_4))=k(\Delta_{G\times G'}\otimes 1)\Delta_{G\times G'}((g,g'))= (\Delta_{G\times G'}\otimes 1)\Delta_{G\times G'}(k(g,g')).$$

We complete the coassociation law. The next equality following directly from the definition of counit.

$$(1\otimes\varepsilon_{G\times G'})\Delta_{G\times G'}(k(g,g'))= k\sum_{\substack{g_1g_2=g\\ g'_1g'_2=g'}}\Delta_{g_2,e_G}\Delta_{g'_2,e_{G'}}(g_1,g_1')=k(g,g').$$

It is similar to show that $(\varepsilon_{G\times G'}\otimes 1)\Delta_{G\times G'}(k(g,g'))=k(g,g')$.

Lemma 2.2 $K[G]\otimes K[G']$ is grouplike coalgebra defined by

$$\Delta_{G\otimes G'}((g\otimes g'))=\sum_{\substack{g_1g_2=g\\ g'_1g'_2=g'}}((g_1\otimes g'_1)\otimes (g_2\otimes g_2')), \varepsilon_{G\otimes G'}((g\otimes g'))=\delta_{g,e_G}\delta_{g',e_{G'}}.$$

Proof Now we show the coassociation law. For any $kg\otimes lg'\in K[G]\otimes K[G'], k,l\in K, g\in G, g'\in G'$,

$$(1\otimes\Delta_{G\otimes G'})\Delta_{G\otimes G'}(kg\otimes lg')=kl(1\otimes \Delta_{G\otimes G'})\Delta_{G\otimes G'}((g\otimes g'))=kl(1\otimes \Delta_{G\otimes G'})(\sum_{\substack{g_1g_2=g\\ g'_1g'_2=g'}}(g_1\otimes g_1')\otimes(g_2\otimes g_2'))= kl\sum_{\substack{g_1g_2=g\\ g'_1g'_2=g'}}((g_1\otimes g'_1)\otimes\Delta_{G\otimes G'}((g_2\otimes g'_2)))= kl\sum_{\substack{g_1g_2=g\\ g'_1g'_2=g'\\ g_3g_4=g_2\\ g'_3g'_4=g'_2}}(g_1\otimes g'_1\otimes g_3\otimes g'_3\otimes g_4\otimes g'_4)= kl\sum_{\substack{g_1g_3=g_5\\ g'_1g'_3=g'_5\\ g_5g_4=g\\ g'_5g'_4=g'}}(\Delta_{G\otimes G'}((g_5\otimes g_5'))\otimes(g_4\otimes g_4'))= kl(\Delta_{G\otimes G'}\otimes 1)(\sum_{\substack{g_5g_4=g\\ g'_5g'_4=g'}}(g_5\otimes g_5')\otimes(g_4\otimes g_4'))= kl(\Delta_{G\otimes G'}\otimes 1)\Delta_{G\otimes G'}((g\otimes g'))=(\Delta_{G\otimes G'}\otimes 1)\Delta_{G\otimes G'}(kg\otimes lg')).$$

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Next, by the properties of $\varepsilon_{G\otimes G'}$, we can easily get

$$(1\otimes \varepsilon_{G\otimes G'})\Delta_{G\otimes G'}(kg\otimes lg'))= kl\sum_{\substack{g_1g_2=g\\ g'_1g'_2=g'}}\delta_{g_2,e_G}\delta_{g'_2,e_{G'}}(g_1\otimes g'_1)=kl(g\otimes g')=(kg\otimes lg').$$

In the same way, we can show that $(\varepsilon_{G\otimes G'}\otimes 1)\Delta_{G\otimes G'}(kg\otimes lg')=(kg\otimes lg')$.

A similar result also holds when we cosider monoid as set. Therefore we can obtain the following inclusion immediately.

Theorem 2.1 The K-linear isomorphism $K[G\times G']\simeq K[G]\otimes K[G']$ defined by $f'((g,g'))=g\otimes g'$, for all $g\in G, g'\in G'$ is an isomorphism of coalgebra.

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