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一个广义薄膜方程扰动的有限传播

A Finite Propagation of Perturbations of Weak Solutions for the Generalized Thin Film Equation

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摘要:在一维空间中,利用能量等式、Hardy 不等式、Nirenberg 不等式,讨论一个广义薄膜方程的初边值问题扰动的有限传播,得到方程解的支集传播的有限性.

关键词:薄膜方程 弱解 扰动 有限传播

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Abstract: We consider an initial-boundary value problem for the generalized thin film equation in one space dimension. A finite propagation of perturbations of weak solutions is discussed by using the energy equality, Hardy inequality and Nirenberg inequality.

Key words: thin film equation, weak solution, perturbations, finite propagation

考虑如下广义薄膜方程的初边值问题:

$$\frac{\partial u}{\partial t}+\operatorname{div}(|\nabla\triangle u|^{p-2}\nabla\Delta u)-\Delta u=0, x\in\Omega, t>0, p>2, \tag{1}$$

$$u=\Delta u=0, x\in\partial\Omega, t>0, \tag{2}$$

$$u(x,0)=u_0(x), x\in\Omega, \tag{3}$$

其中 $\Omega\subset R^N$ 为具光滑边界的有界区域.

方程(1)为典型的高阶方程,具有很强的物理背景和丰富的理论内涵,它描述了在近似光滑的固体表面上,由表面张力推动的液体薄膜高度 $u(x,t)$ 的演变[1,2]. J. R. King 导出方程(1)后,又利用支集度的局部分析方法研究了一维情形下的 Cauchy 问题和特殊闭形解,如行波解、可分离解、瞬时源解等. 在二维情形下构造整个固体表面上油膜扩散模型[3]时,也可以用方程(1)来表示. 关于广义薄膜方程的初边值问题,文献[4]已经证明其弱解的存在性. 文献[5]进一步证明弱解的唯一性并讨论了渐近行为. 而最近,文献[6]和[7]在讨论一类重要的拟线性退化抛物方程——非牛顿渗流方程的逼近能控性时,发现利用解的有限传播速度可以说明如果对系统施加局部控制,即限制控制的作用在所考虑区域的一个非空真开子集上,那么对于任何初值,系统在短时间内都不是逼近能控的.

基于此,本文在一维空间中,利用能量等式、Hardy 不等式、Nirenberg 不等式,讨论问题(1)～(3)弱解扰动的有限传播速度. 为方便,假定 $\Omega=(0,1)$.

定理1 对任意的 $\rho(x)\geqslant 0, \rho(x)\in C^2(\overline{\Omega})$,问题(1)～(3)的弱解 u 满足

$$\frac{1}{2}\int_0^1\rho(x)\,|Du(x,t)|^2\mathrm{d}x-\frac{1}{2}\int_0^1\rho(x)\,|Du_0(x)|^2\mathrm{d}x=-\iint_{Q_t}|D^3u|^{p-2}D^3uD^2(\rho(x)Du)\mathrm{d}x\mathrm{d}\tau-\iint_{Q_t}D^2uD(\rho(x)Du)\mathrm{d}x\mathrm{d}\tau, \tag{4}$$

其中,$Q_t=(0,1)\times(0,t), D=\frac{\partial}{\partial x}$.

证明 类似于参考文献[4]中定理2.1的证明,对任意的 $0\leqslant\rho\in C^2(\overline{\Omega})$,有

$$f_\rho(t)=\frac{1}{2}\int_\Omega\rho(x)\,|Du(x,t)|^2\mathrm{d}x\in C([0,T]).$$

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考虑泛函

$$\Phi_\rho[v]=\frac{1}{2}\int_\Omega \rho(x)\mid Dv(x)\mid^2 dx.$$

容易看出,$\Phi_\rho[v]$ 是 $H_0^1(\Omega)$ 上的凸泛函. 而对任意的 $\tau \in (0,T)$ 和 $h>0$,有

$$\Phi_\rho[u(\tau+h)]-\Phi_\rho[u(\tau)]\geqslant\langle u(\tau+h)-u(\tau), -D(\rho(x)D\,u(\tau))\rangle.$$

又由 $\frac{\delta\Phi_\rho[v]}{\delta v}=-D(\rho(x)D\,v)$,再对任意给定的 $t_1,t_2 \in [0,T]$,$t_1<t_2$,上式关于 τ 在 (t_1,t_2) 上积分,得

$$\int_{t_2}^{t_2+h}\Phi_\rho[u(\tau)]d\tau-\int_{t_1}^{t_1+h}\Phi_\rho[u(\tau)]d\tau\geqslant\int_{t_1}^{t_2}\langle u(\tau+h)-u(\tau),-D(\rho(x)D\,u(\tau))\rangle\,d\tau.$$

在上式两边乘以 $\frac{1}{h}$,并令 $h\to 0$,得到

$$\Phi_\rho[u(t_2)]-\Phi_\rho[u(t_1)]\geqslant\int_{t_1}^{t_2}\langle\frac{\partial u}{\partial t},-D(\rho(x)Du(\tau))\rangle\,d\tau.$$

类似地,有

$$\Phi_\rho[u(\tau)]-\Phi_\rho[u(\tau-h)]\leqslant\langle u(\tau)-u(\tau-h),-D(\rho(x)D\,u(\tau))\rangle.$$

从而

$$\Phi_\rho[u(t_2)]-\Phi_\rho[u(t_1)]\leqslant\int_{t_1}^{t_2}\langle\frac{\partial u}{\partial t},-D(\rho(x)Du(\tau))\rangle\,d\tau,$$

因此

$$\Phi_\rho[u(t_2)]-\Phi_\rho[u(t_1)]=\int_{t_1}^{t_2}\langle\frac{\partial u}{\partial t},-D(\rho(x)D\,u(\tau))\rangle\,d\tau.$$

取 $t_1=0,t_2=t$,由弱解的定义,得到

$$\Phi_\rho[u(t)]-\Phi_\rho[u(0)]=\int_0^t\langle -D(\mid D^3u\mid^{p-2}D^3u)+D^2u,-D(\rho(x)D\,u(\tau))\rangle\,d\tau=-\int_0^t\langle\mid D^3u\mid^{p-2}D^3u,D^2(\rho(x)D\,u(\tau))\rangle\,d\tau-\int_0^t\langle D^2u,D(\rho(x)Du(\tau))\rangle d\tau.$$

证明完毕.

定理 2 假设 $p>2$,$\operatorname{supp} u_0\subset[x_1,x_2]$,$0<x_1<x_2<1$,且 u 是问题(1)~(3)的弱解,那么对任意固定的 $t>0$,有

$$\operatorname{supp}u(x,\cdot)\subset[x_1(t),x_2(t)]\cap[0,1],$$

其中 $x_1(t)=x_1-C_1t^{\frac{1}{3p-2}}$,$x_2(t)=x_2+C_2t^{\frac{1}{3p-2}}$,

$$C_1=C(\int_0^T\int_0^{x_1}\mid D^3u\mid^p dxd\tau)^{\frac{p-2}{2(3p-2)}},C_2=C(\int_0^T\int_{x_2}^1\mid D^3u\mid^p dxd\tau)^{\frac{p-2}{2(3p-2)}},C=2^{3p-2}(2p+1)^p(p-1)^{p-1}(1+p^{-p}).$$

证明 在(4)式中,取 $\rho(x)=(x-y)_+^s$,$y\in[x_2,1]$,有

$$\frac{1}{2}\int_0^1(x-y)_+^s\mid Du(x,t)\mid^2dx=-\int_0^t\int_0^1\mid D^3u\mid^{p-2}D^3uD^2[(x-y)_+^s\,Du]dxd\tau-\int_0^t\int_0^1D^2uD[(x-y)_+^s\,Du]dxd\tau.$$

记上式的左边为 J,则

$$J=-\int_0^t\int_0^1\mid D^3u\mid^{p-2}D^3uD^2[(x-y)_+^s\,Du]dxd\tau-\int_0^t\int_0^1D^2uD[(x-y)_+^s\,Du]dxd\tau=$$
$$-\int_0^t\int_0^1(x-y)_+^s\mid D^3u\mid^p dxd\tau-2s\int_0^t\int_0^1(x-y)_+^{s-1}D^2u\mid D^3u\mid^{p-2}D^3u\,dxd\tau-s(s-1)\int_0^t\int_0^1(x-y)_+^{s-2}\mid D^3u\mid^{p-2}D^3uDu\,dxd\tau-$$
$$\int_0^t\int_0^1(x-y)_+^s\mid D^2u\mid^2dxd\tau-s\int_0^t\int_0^1(x-y)_+^{s-1}D^2uDu\,dxd\tau.$$

利用 Young 不等式,有

$$J\leqslant-\int_0^t\int_0^1(x-y)_+^s\mid D^3u\mid^p dxd\tau+\frac{1}{4}\int_0^t\int_0^1(x-y)_+^s\mid D^3u\mid^p dxd\tau+C_1\int_0^t\int_0^1(x-y)_+^{s-p}\mid D^2u\mid^p dxd\tau+$$
$$\frac{1}{4}\int_0^t\int_0^1(x-y)_+^s\mid D^3u\mid^p dxd\tau+C_2\int_0^t\int_0^1(x-y)_+^{s-2p}\mid Du\mid^p dxd\tau+C_3\int_0^t\int_0^1(x-y)_+^s\mid D^2u\mid^p dxd\tau+$$
$$C_4\int_0^t\int_0^1(x-y)_+^{s-p}\mid Du\mid^p dxd\tau.$$

由于$(x-y)_+\subset(0,1)$,因此

$$J\leqslant-\frac{1}{2}\int_0^t\int_0^1(x-y)_+^s\mid D^3u\mid^p dxd\tau+C_1\int_0^t\int_0^1(x-y)_+^{s-p}\mid D^2u\mid^p dxd\tau+C_2\int_0^t\int_0^1(x-y)_+^{s-2p}\mid Du\mid^p dxd\tau.$$

再利用 Hardy 不等式[8],得

$$\int_0^1(x-y)_+^{s-2p}\mid Du\mid^p dx\leqslant C\int_0^1(x-y)_+^{s-p}\mid D^2u\mid^p dx.$$

因此

$$\frac{1}{2}\int_0^1(x-y)_+^s\mid Du\mid^2dx+\frac{1}{2}\int_0^t\int_0^1(x-y)_+^s\mid D^3u\mid^p dxd\tau\leqslant C_1\int_0^t\int_0^1(x-y)_+^{s-p}\mid D^2u\mid^p dxd\tau.$$

从而有

$$\sup_{0<\tau\leqslant t}\int_0^1(x-y)_+^s\mid Du\mid^2dx\leqslant C_1\int_0^t\int_0^1(x-y)_+^{s-p}\mid D^2u\mid^p dxd\tau,\tag{5}$$

$$\int_0^t\int_0^1 (x-y)_+^s |D^3u|^p dxd\tau \leqslant C_1\int_0^t\int_0^1 (x-y)_+^{s-p} |D^2u|^p dxd\tau. \qquad (6)$$

对(5)式再次利用 Hardy 不等式,有

$$\sup_{0<\tau\leqslant t}\int_0^1 (x-y)_+^s |Du|^2 dx \leqslant C\int_0^t\int_0^1 (x-y)_+^s |D^3u|^p dxd\tau. \qquad (7)$$

令

$$E_s(y)=\int_0^t\int_0^1 (x-y)_+^s |D^3u|^p dxd\tau, E_0(y)=\int_0^t\int_0^1 |D^3u|^p dxd\tau.$$

在(6)中取 $s=2p+1$,并应用加权 Nirenberg 不等式[9],则有

$$E_{2p+1}(y)\leqslant C_1\iint_{Q_t}(x-y)_+^{p+1} |D^2u|^p dxd\tau \leqslant C\int_0^t\left(\int_0^1 (x-y)_+^{p+1} |D^3u|^p dx\right)^a\left(\int_0^1 (x-y)_+^{p+1} |Du|^2 dx\right)^{(1-a)p/2} d\tau,$$

其中$\frac{1}{p}=\frac{1}{p+2}+a\left(\frac{1}{p}-\frac{2}{p+2}\right)+(1-a)\frac{1}{2}$,因此

$$0<a=\frac{\frac{1}{p}-\frac{1}{p+2}-\frac{1}{2}}{\frac{1}{p}-\frac{2}{p+2}-\frac{1}{2}}<1.$$

再由(7)式和 Hölder 不等式,得到

$$E_{2p+1}(y)\leqslant C\left(\iint_{Q_t}(x-y)_+^{p+1} |D^3u|^p dxd\tau\right)^{(1-a)p/2}\int_0^t\left(\int_0^1 (x-y)_+^{p+1} |D^3u|^p dx\right)^a d\tau \leqslant C[E_{p+1}(y)]^{(1-a)p/2}\left(\iint_{Q_t}(x-y)_+^{p+1} |D^3u|^p dxd\tau\right)^a t^{1-a} \leqslant C(E_{p+1}(y))^{(1-a)p/2+a}t^{1-a}.$$

记 $\lambda=1-a,\gamma=a+(1-a)p/2$,那么由 Hölder 不等式,有

$$E_{2p+1}(y)\leqslant Ct^\lambda\left[\iint_{Q_t}(x-y)_+^{p+1} |D^3u|^p dxd\tau\right]^\gamma \leqslant Ct^\lambda\left[\iint_{Q_t}(x-y)_+^{2p+1} |D^3u|^p dxd\tau\right]^{\frac{(p+1)\gamma}{2p+1}}\left[\int_0^t\int_y^1 |D^3u|^p dxd\tau\right]^{\frac{p\gamma}{2p+1}} \leqslant Ct^\lambda[E_{2p+1}(y)]^{(p+1)\gamma/(2p+1)}[E_0(y)]^{p\gamma/(2p+1)}.$$

因此

$$E_{2p+1}(y)\leqslant Ct^{\lambda/\sigma}[E_0(y)]^{p\gamma/((2p+1)\sigma)}, \sigma=1-\frac{p+1}{2p+1}\gamma>0.$$

再次利用 Hölder 不等式,得

$$E_1(y)\leqslant[E_{2p+1}(y)]^{1/(2p+1)}[E_0(y)]^{2p/(2p+1)}\leqslant Ct^{\gamma_1}[E_0(y)]^{1+\theta},$$

其中

$$\gamma_1=\frac{\lambda}{\sigma(2p+1)}, \theta=\frac{p\gamma}{\sigma(2p+1)^2}-\frac{1}{2p+1}>0.$$

注意到 $E_1'(y)=-E_0(y)$,则

$$E_1'(y)\leqslant -Ct^{-\gamma_1/(\theta+1)}[E_1(y)]^{1/(\theta+1)}.$$

若 $E_1(x_2)=0$,则$\mathrm{supp}u\subset[0,x_2]$. 若 $E_1(x_2)>0$,则存在一个最大区间$[x_2,x_2^*]$,使得 $E_1(y)>0$, $E_1(x_2^*)=0$,且

$$[(E_1(y))^{\theta/(\theta+1)}]'=\frac{\theta}{\theta+1}\frac{E_1'(y)}{[E_1(y)]^{\theta/(\theta+1)}}\leqslant -Ct^{-\gamma_1/(\theta+1)}.$$

把上式在(x_2,x_2^*)上积分,得

$$E_1(x_2^*)^{\theta/(\theta+1)}-E_1(x_2)^{\theta/(\theta+1)}\leqslant -Ct^{-\gamma_1/(\theta+1)}(x_2^*-x_2),$$

即

$$x_2^*\leqslant x_2+Ct^\mu(E_0(x_2))^{\theta/(\theta+1)}\equiv x_2(t), \mu=\frac{\gamma_1}{\theta+1}=\frac{1}{3p-2}>0.$$

再由参考文献[4]的结果知,$E_0(y)$被一个不依赖于 y 的常数 C 所控制. 所以

$$\mathrm{supp}u(\cdot,t)\subset[0,x_2(t)].$$

类似地,可以证明

$$\mathrm{supp}u(\cdot,t)\subset[x_1(t),1].$$

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