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# 交叉余积上的弱 Hopf 代数结构*

# The Weak Hopf Algebra Structure on Crossed Coproduct

任北上,夏嘉艺,林仕勋,郭映雪

REN Bei-shang,XIA Jia-yi,LIN Shi-xun,GUO Ying-xue

(广西师范学院数学科学学院,广西南宁 530023)

(College of Mathematical Sciences,Guangxi Teachers Education University,Nanning,Guangxi,530023,China)

**摘要**:利用左 $H$-模、弱双代数、交叉积等工具给出了交叉余积成为弱双代数与 Hopf 代数的充要条件.

**关键词**:交叉余积 弱左余模 弱双代数 弱 Hopf 代数

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**Abstract**:We use the concept of weak left $H$ comodule,weak bialgebra and crossed produut to discuss the necessary and sufficient condition for the crossed coproduct that is weak bialgebra and weak Hopf algebra.

**Key words**:crossed coproduct,weak left comodule,weak bialgebra, weak Hopf algebra

近 30 年来,弱 Hopf 代数以更宽泛的视角和独特的内涵,给 Hopf 代数的研究增添了活力,国内的不少学者正是通过弱化条件构造了多个不同结构的弱 Hopf 代数.文献[1]引进了交叉积的对偶交叉余积的概念,证明了余 cleft 模余代数的结构定理,讨论了 Hopf 代数余可裂正合序列并推广了余代数的结构性质.本文在此基础上探讨交叉余积上弱 Hopf 代数的结构问题,所有符号与文献[1,2]保持一致,所有向量空间默认为域 $K$ 上的,$H$ 为 Hopf 代数,$A$ 为代数.

## 1 基本定义和引理

**定义 1.1**[2] 设 $H$ 是 Hopf 代数,$A$ 为代数.若 $H$ 度量 $A$ 且 $\sigma \in \mathrm{Hom}_K(H \otimes H, A)$ 是可逆元,则 $A$ 关于 $H$ 的交叉积 $A \#_\sigma H$ 是 $A \otimes H$,作为向量空间乘法为

$$(a \# h)(b \# k) = \sum (h_1 \cdot a)\sigma(h_2, k_1) \# h_3 k_2,$$

其中 $h,k \in H, a,b \in A$.

这里,把 $a \otimes h$ 写作 $a \# h$.当 $\sigma$ 是平凡的,即 $\sigma(h,k) = \epsilon(h)\epsilon(k)1_A$ 时,记 $A \#_\sigma H$ 为 $A_\sigma[H]$,且称其交叉积为扭转积,乘法为

$$(a \# h)(b \# k) = \sum a(h_1 \cdot b) \# h_3 k,$$

其中 $h,k \in H, a,b \in A$.

**定义 1.2**[1] 设 $C$ 是弱左 $H$-余模,线性映射 $\alpha$:$C \to H \otimes H$,其中 $\alpha(c) = \sum \alpha_1(c) \otimes \alpha_2(c)$,$\forall c \in C$.向量空间 $C \times_\alpha H = C \otimes H$ 关于余乘

$$\Delta(c \times h) = \sum c_1 \times c_2^1 \alpha_1(c_3) h_1 \otimes c_2^2 \times \alpha_2(c_3) h_2,$$

且 $\rho(c) = \sum c^1 \otimes c^2$ 为左 $H$-余模结构映射,$\forall c \in C, \forall h \in H$.

我们记 $c \otimes h$ 为 $c \times h$,如果 $\varepsilon(c \times h) = \varepsilon_C(c)\varepsilon_H(h)$ 是余单位且满足余结合律,则称 $C \times_\alpha H$ 为关于 $\rho$ 和 $\alpha$ 的交叉余积.若 $\rho$ 是平凡的,即 $\rho(c) = 1 \otimes c$,则记 $C \times_\alpha H$ 为 $C_\alpha[H]$,并称其余交叉余积为扭转余积,余乘为

$$\Delta(c \times h) = \sum c_1 \times \alpha_1(c_3) h_1 \otimes c_2 \times \alpha_2(c_3) h_2,$$

其中 $\forall c \in C, \forall h \in H$.

**引理 1.1**[2] $A \#_\sigma H$ 是结合代数,其单位元为 $1 \# 1$,当且仅当

(1) $A$ 是扭转 $H$-模,即 $1 \cdot a = a$,且

$$h \cdot (k \cdot a) = \sum \sigma(h_1, k_1)(h_2 k_2 \cdot a)\sigma^{-1}(h_3, k_3),$$

其中 $h,k \in H, a \in A$.

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作者简介:任北上(1956-),男,副教授,主要从事环论、模、Hopf 代数方面的研究。

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(2)$\sigma$ 是余循环，即 $\sigma(h,1)=\sigma(1,h)=\varepsilon(h)1$，且 $\sum[h_1\cdot\sigma(k_1,m_1)]\sigma(h_2,k_2m_2)=\sum\sigma(h_1,k_1)\sigma(h_2k_2,m)$，
其中 $h,k,m\in H$.

若 $A$ 不必是 $H$-模且 $\sigma$ 的值也不一定在 $A$ 的中心，那么由引理 1.1 的对偶性，有

**引理 1.2**[1] 如果定义 1.2 中的 $C\times_\alpha H$ 对 $\forall c\in C$，有 $\sum\varepsilon(\alpha_2(c))\alpha_1(c)=\varepsilon(c)1_H$，那么 $C\times_\alpha H$ 是余结合余代数当且仅当，对 $\forall c\in C$，

(1) $\sum c_1^1\alpha_1(c_2)\otimes\alpha_1(c_1^2)(\alpha_2(c_2))_1\otimes\alpha_2(c_1^2)(\alpha_2(c_2))_2=\sum\alpha_1(c_1)(\alpha_1(c_2))_1\otimes\alpha_2(c_1)(\alpha_1(c_2))_2\otimes\alpha_2(c_2)$.

(2) $\sum c_1^1\alpha_1(c_2)\otimes c_1^{21}\alpha_2(c_2)\otimes c_1^{22}=\sum\alpha_1(c_1)(c_2^1)_1\otimes\alpha_2(c_1)(c_2^1)_2\otimes c_2^2$.

此时，我们称 $C$ 为左 $H$-余模余代数.

由文献[3]知，当 $H$ 是弱 Hopf 代数，那么

(1) 设 $C$ 是余代数. 如果存在 $K$-线性映射 $\alpha: C\to H\otimes H$ 使得 $\alpha(c)=\sum\alpha_1(c)\otimes\alpha_2(c)$，那么 $\alpha$ 是可逆映射并满足

(a) $\alpha_1(c)\otimes\varepsilon_H(\alpha_2(c)_1h)\alpha_2(c)_2=\varepsilon_H(\alpha_1(c)_2h)\alpha_1(c)_1\otimes\alpha_2(c)$,

(b) $\alpha_1(c)\otimes\varepsilon_H(h\alpha_2(c)_1)\alpha_2(c)_2=\varepsilon_H(h\alpha_1(c)_2)\alpha_1(c)_1\otimes\alpha_2(c)$,

其中 $\forall c\in C,\forall h\in H$.

(2) 设 $C$ 是弱双代数. 若 $\alpha\in\mathrm{Hom}(C,H\otimes H)$，对 $\forall c\in C$，则

$1_1\alpha_1(c)\otimes1_2\alpha_2(c)=\alpha_1(c)\otimes\alpha_2(c)=\alpha_1(c)1_1\otimes\alpha_2(c)1_2$.

## 2 主要结果

**引理 2.1** 设 $C$ 是弱双代数，$H$ 是弱 Hopf 代数，则 $C\times_\alpha H$ 是余结合余代数，当且仅当，对 $\forall c\in C$

(1) $\sum c_1^1\alpha_1(c_2)\otimes\alpha_1(c_1^2)(\alpha_2(c_2))_1\otimes\alpha_2(c_1^2)(\alpha_2(c_2))_2=\sum\alpha_1(c_1)(\alpha_1(c_2))_1\otimes\alpha_2(c_1)(\alpha_1(c_2))_2\otimes\alpha_2(c_2)$,

(2) $\sum c_1^1\alpha_1(c_2)\otimes c_1^{21}\alpha_2(c_2)\otimes c_1^{22}=\sum\alpha_1(c_1)(c_2^1)_1\otimes\alpha_2(c_1)(c_2^1)_2\otimes c_2^2$.

**引理 2.2** 设 $C$ 是弱双代数，$H$ 是弱 Hopf 代数，$C\times_\alpha H$ 为交叉余积，则 $\Delta_{C\times_\alpha H}$ 是余乘映射，当且仅当

(1) $\sum a_1b_1\alpha(a_2b_2)=\sum a_1\alpha_1(a_2)b_1\alpha_1(b_2)\otimes\alpha_2(a_2)\alpha_2(b_2)$,

(2) $\sum b_1\alpha_1(b_2)k_1\otimes\alpha_2(b_2)k_2=\sum k_1b_1\alpha_1(b_2)\otimes k_2\alpha_2(b_2)$.

其中，$C\times_\alpha H$ 的乘法是 $(a\times k)(b\times h)=ab\times kh$，$\forall a,b\in C,\forall h,k\in H$.

**证明** 由于 $\Delta_{C\times_\alpha H}((a\times k)(b\times h))=\Delta_{C\times_\alpha H}(a\times k)\Delta_{C\times_\alpha H}(b\times h)$，其中

$\Delta_{C\times_\alpha H}((a\times k)(b\times h))=\Delta_{C\times_\alpha H}(ab\times kh)=\sum a_1b_1\times a_2^1b_2^1\alpha_1(a_3b_3)k_1h_1\otimes a_2^2b_2^2\times\alpha_2(a_3b_3)k_2h_2$,

$\Delta_{C\times_\alpha H}(a\times k)\Delta_{C\times_\alpha H}(b\times h)=\sum(a_1\times a_2^1\alpha_1(a_3)k_1\otimes a_2^2\times\alpha_2(a_3)k_2)(b_1\times b_2^1\alpha_1(b_3)h_1\otimes b_2^2\times\alpha_2(b_3)h_2)=\sum a_1b_1\times a_2^1\alpha_1(a_3)k_1b_2^1\alpha_1(b_3)h_1\otimes a_2^2b_2^2\times\alpha_2(a_3)k_2\alpha_2(b_3)h_2$,

故有

$$\sum a_1b_1\times a_2^1b_2^1\alpha_1(a_3b_3)k_1h_1\otimes a_2^2b_2^2\times\alpha_2(a_3b_3)k_2h_2=\sum a_1b_1\times a_2^1\alpha_1(a_3)k_1b_2^1\alpha_1(b_3)h_1\otimes a_2^2b_2^2\times\alpha_2(a_3)k_2\alpha_2(b_3)h_2.$$

把 $(\varepsilon\times I\otimes\varepsilon\times I)$ 作用于上式两边，得

$$\sum\varepsilon(a_1b_1)\times a_2^1b_2^1\alpha_1(a_3b_3)k_1h_1\otimes\varepsilon(a_2^2b_2^2)\times\alpha_2(a_3b_3)k_2h_2=\sum\varepsilon(a_1b_1)\times a_2^1\alpha_1(a_3)k_1b_2^1\alpha_1(b_3)h_1\otimes\varepsilon(a_2^2b_2^2)\times\alpha_2(a_3)k_2\alpha_2(b_3)h_2,$$

$$\sum a_1b_1\alpha_1(a_2b_2)k_1h_1\otimes\alpha_2(a_2b_2)k_2h_2=\sum a_1\alpha_1(a_2)k_1b_1\alpha_1(b_2)h_1\otimes\alpha_2(a_2)k_2\alpha_2(b_2)h_2.$$

若令 $k=1,h=1$，有

$$\sum a_1b_1\alpha(a_2b_2)=\sum a_1\alpha_1(a_2)b_1\alpha_1(b_2)\otimes\alpha_2(a_2)\alpha_2(b_2).$$

若令 $a=1,h=1$，有

$$\sum b_1\alpha_1(b_2)k_1\otimes\alpha_2(b_2)k_2=\sum k_1b_1\alpha_1(b_2)\otimes k_2\alpha_2(b_2).$$

**引理 2.3** 设 $C$ 是弱双代数，$H$ 是弱 Hopf 代数，$C\times_\alpha H$ 为交叉余积，则 $\varepsilon_{C\times_\alpha H}$ 是余单位映射，当且仅当

(1) $\sum\varepsilon_C(b)\varepsilon_H(khp)=\sum\varepsilon_C(b_1b_2^2)\varepsilon_H(kb_2^1\alpha_1(b_3)h_1)\varepsilon_H(\alpha_2(b_3)h_2p)$,

(2) $\sum\varepsilon_C(b)\varepsilon_H(khp)=\sum\varepsilon_C(b_2^2b_1)\varepsilon_H(k\alpha_2(b_3)h_2)\varepsilon_H(b_2^1\alpha_1(b_3)h_1p)$,

其中，$\forall a,b,c\in C,\forall k,h,p\in H$.

**证明** 由弱 Hopf 代数定义可知，

$\varepsilon_{C\times_\alpha H}((a\times k)(b\times h)(c\times p))=\sum\varepsilon_{C\times_\alpha H}((a\times k)(b\times h)_1)\varepsilon_{C\times_\alpha H}((b\times h)_2(c\times p))$.

那么，有

$\varepsilon_{C\times_\alpha H}((a\times k)(b\times h)(c\times p))=\sum\varepsilon_{C\times_\alpha H}(abc\times khp)=\sum\varepsilon_C(abc)\varepsilon_H(khp)$,

$$\sum\varepsilon_{C\times_\alpha H}((a\times k)(b\times h)_1)\varepsilon_{C\times_\alpha H}((b\times h)_2(c\times p))=\sum\varepsilon_{C\times_\alpha H}((a\times k)(b_1\times b_2^1\alpha_1(b_3)h_1))\varepsilon_{C\times_\alpha H}((b_2^2\times\alpha_2(b_3)h_2)(c\times p))=$$
$$\sum\varepsilon_C(ab_1b_2^2c)\varepsilon_H(kb_2^1\alpha_1(b_3)h_1)\varepsilon_H(\alpha_2(b_3)h_2p).$$

**令 $a=1,c=1$,则有**

$$\sum\varepsilon_C(b)\varepsilon_H(khp)=\sum\varepsilon_C(b_1b_2^2)\varepsilon_H(kb_2^1\alpha_1(b_3)h_1)\varepsilon_H(\alpha_2(b_3)h_2p).$$

**类似可得,**

$$\sum\varepsilon_C(b)\varepsilon_H(khp)=\sum\varepsilon_C(b_2^2b_1)\varepsilon_H(k\alpha_2(b_3)h_2)\varepsilon_H(b_2^1\alpha_1(b_3)h_1p).$$

**引理 2.4** 设 $C$ 是弱双代数,$H$ 是弱 Hopf 代数,$C\times_\alpha H$ 为交叉余积,则以下等式成立

(1) $\sum 1_{C_1}^1\alpha_1(1_{C_4})\otimes 1_{C_2}^1\alpha_1(1_{C_3}^2)\alpha_2(1_{C_4})_1\otimes 1_{C_2}\otimes\alpha_2(1_{C_3}^2)\alpha_2(1_{C_4})_2=\sum 1_{C_3}\alpha_1(1_{C_4})\otimes 1_{C_1}^1\alpha_1(1_{C_2})\alpha_2(1_{C_4})\otimes 1_{C_1}^2\otimes\alpha_2(1_{C_2})$.

(2) $\sum 1_{C_1}^1\alpha_1(1_{C_4})\otimes 1_{C_2}^1\alpha_1(1_{C_3}^2)\alpha_2(1_{C_4})_1\otimes 1_{C_2}\otimes\alpha_2(1_{C_3}^2)\alpha_2(1_{C_4})_2=\sum 1_{C_1}\alpha_1(1_{C_2})\otimes\alpha_2(1_{C_2})1_{C_3}^1\alpha_1(1_{C_4})\otimes 1_{C_3}^2\otimes\alpha_2(1_{C_4})$.

**证明** 由弱 Hopf 代数定义可知

$$(I\otimes\Delta_{C\times_\alpha H})\Delta_{C\times_\alpha H}(1_C\times 1_H)=((1_C\times 1_H)\otimes\Delta_{C\times_\alpha H}(1_C\times 1_H))(\Delta_{C\times_\alpha H}(1_C\times 1_H)\otimes(1_C\times 1_H)).$$

**那么,有**

$$(I\otimes\Delta_{C\times_\alpha H})\Delta_{C\times_\alpha H}(1_C\times 1_H)=\sum(I\otimes\Delta_{C\times_\alpha H})(1_{C_1}\times 1_{C_2}^1\alpha_1(1_{C_3})1_{H_1}\otimes 1_{C_2}^2\times\alpha_2(1_{C_3})1_{H_2})=\sum 1_{C_1}\times 1_{C_2}^1\alpha_1(1_{C_3})1_{H_1}\otimes\Delta(1_{C_2}^2\times\alpha_2(1_{C_3})1_{H_2})=\sum 1_{C_1}\times 1_{C_2}^1\alpha_1(1_{C_5})\otimes 1_{C_2}\times 1_{C_3}^1\alpha_1(1_{C_4}^2)\alpha_2(1_{C_5})_1\otimes 1_{C_3}\times\alpha_2(1_{C_4}^2)\alpha_2(1_{C_5})_2,$$

$$((1_C\times 1_H)\otimes\Delta_{C\times_\alpha H}(1_C\times 1_H))(\Delta_{C\times_\alpha H}(1_C\times 1_H)\otimes(1_C\times 1_H))=\sum(1_C\times 1_H\otimes 1_{C_1}\times 1_{C_2}^1\alpha_1(1_{C_3})1_{H_1}\otimes 1_{C_2}^2\times\alpha_2(1_{C_3})1_{H_2})(1_{C_4}\times 1_{C_5}^1\alpha_1(1_{C_6})1_{H_3}\otimes 1_{C_5}^2\times\alpha_2(1_{C_6})1_{H_4})=\sum 1_{C_4}\times 1_{C_5}^1\alpha_1(1_{C_6})1_{H_3}\otimes 1_{C_1}1_{C_5}^2\times 1_{C_2}^1\alpha_1(1_{C_3})1_{H_1}\alpha_2(1_{C_6})1_{H_4}\otimes 1_{C_2}^2\times\alpha_2(1_{C_3})1_{H_2}=\sum 1_{C_4}\times 1_{C_5}^1\alpha_1(1_{C_6})\otimes 1_{C_1}1_{C_5}^2\times 1_{C_2}^1\alpha_1(1_{C_3})\alpha_2(1_{C_6})\otimes 1_{C_2}^2\times\alpha_2(1_{C_3}).$$

**故得**

$$\sum 1_{C_1}\times 1_{C_2}^1\alpha_1(1_{C_5})\otimes 1_{C_2}\times 1_{C_3}^1\alpha_1(1_{C_4}^2)\alpha_2(1_{C_5})_1\otimes 1_{C_3}\times\alpha_2(1_{C_4}^2)\alpha_2(1_{C_5})_2=\sum 1_{C_4}\times 1_{C_5}^1\alpha_1(1_{C_6})\otimes 1_{C_1}1_{C_5}^2\times 1_{C_2}^1\alpha_1(1_{C_3})\alpha_2(1_{C_6})\otimes 1_{C_2}^2\times\alpha_2(1_{C_3}).$$

**把 $\varepsilon\times I\otimes\varepsilon\times I\otimes I_{C\times_\alpha H}$ 作用于等式两端,则**

$$\sum 1_{C_1}^1\alpha_1(1_{C_4})\otimes 1_{C_2}^1\alpha_1(1_{C_3}^2)\alpha_2(1_{C_4})_1\otimes 1_{C_2}\otimes\alpha_2(1_{C_3}^2)\alpha_2(1_{C_4})_2=\sum 1_{C_3}\alpha_1(1_{C_4})\otimes 1_{C_1}^1\alpha_1(1_{C_2})\alpha_2(1_{C_4})\otimes 1_{C_1}^2\otimes\alpha_2(1_{C_2}).$$

**类似可得**

$$\sum 1_{C_1}^1\alpha_1(1_{C_4})\otimes 1_{C_2}^1\alpha_1(1_{C_3}^2)\alpha_2(1_{C_4})_1\otimes 1_{C_2}\otimes\alpha_2(1_{C_3}^2)\alpha_2(1_{C_4})_2=\sum 1_{C_1}\alpha_1(1_{C_2})\otimes\alpha_2(1_{C_2})1_{C_3}^1\alpha_1(1_{C_4})\otimes 1_{C_3}^2\otimes\alpha_2(1_{C_4}).$$

**定理 2.1** 设 $C$ 是弱双代数,$H$ 是弱 Hopf 代数,则交叉余积 $C\times_\alpha H$ 为弱双代数,当且仅当

(1) $\sum c_1^1\alpha_1(c_2)\otimes\alpha_1(c_1^2)(\alpha_2(c_2))_1\otimes\alpha_2(c_1^2)(\alpha_2(c_2))_2=\sum\alpha_1(c_1)(\alpha_1(c_2))_1\otimes\alpha_2(c_1)(\alpha_1(c_2))_2\otimes\alpha_2(c_2)$,

(2) $\sum c_1^1\alpha_1(c_2)\otimes c_1^{21}\alpha_2(c_2)\otimes c_1^{22}=\sum\alpha_1(c_1)(c_2^1)_1\otimes\alpha_2(c_1)(c_2^1)_2\otimes c_2^2$,

(3) $\sum a_1b_1\alpha(a_2b_2)=\sum a_1\alpha_1(a_2)b_1\alpha_1(b_2)\otimes\alpha_2(a_2)\alpha_2(b_2)$,

(4) $\sum b_1\alpha_1(b_2)k_1\otimes\alpha_2(b_2)k_2=\sum k_1b_1\alpha_1(b_2)\otimes k_2\alpha_2(b_2)$,

(5) $\sum\varepsilon_C(b)\varepsilon_H(khp)=\sum\varepsilon_C(b_1b_2^2)\varepsilon_H(kb_2^1\alpha_1(b_3)h_1)\varepsilon_H(\alpha_2(b_3)h_2p)$,

(6) $\sum\varepsilon_C(b)\varepsilon_H(khp)=\sum\varepsilon_C(b_2^2b_1)\varepsilon_H(k\alpha_2(b_3)h_2)\varepsilon_H(b_2^1\alpha_1(b_3)h_1p)$,

(7) $\sum 1_{C_1}^1\alpha_1(1_{C_4})\otimes 1_{C_2}^1\alpha_1(1_{C_3}^2)\alpha_2(1_{C_4})_1\otimes 1_{C_2}\otimes\alpha_2(1_{C_3}^2)\alpha_2(1_{C_4})_2=\sum 1_{C_3}\alpha_1(1_{C_4})\otimes 1_{C_1}^1\alpha_1(1_{C_2})\alpha_2(1_{C_4})\otimes 1_{C_1}^2\otimes\alpha_2(1_{C_2})$,

(8) $\sum 1_{C_1}^1\alpha_1(1_{C_4})\otimes 1_{C_2}^1\alpha_1(1_{C_3}^2)\alpha_2(1_{C_4})_1\otimes 1_{C_2}\otimes\alpha_2(1_{C_3}^2)\alpha_2(1_{C_4})_2=\sum 1_{C_1}\alpha_1(1_{C_2})\otimes\alpha_2(1_{C_2})1_{C_3}^1\alpha_1(1_{C_4})\otimes 1_{C_3}^2\otimes\alpha_2(1_{C_4})$.

**已知 $\alpha$ 是可逆映射,且 $\varepsilon_c(c)1_H=\sum S(\alpha_1(c))\alpha_2(c)=\sum\alpha_1(c)S(\alpha_2(c))$,故有 $\varepsilon_c(c)1_H=\sum S(\alpha_1{}^{-1}(c))\alpha_2{}^{-1}(c)=\sum\alpha_1{}^{-1}(c)S(\alpha_2{}^{-1}(c))$.**

**引理 2.5** 设 $C$ 是弱双代数,$H$ 是弱 Hopf 代数,交

叉余积 $C\times_\alpha H$ 为弱双代数，若映射 $\alpha$ 可逆，则有

$$\sum c_1^1\alpha_1^{-1}(c_2)\otimes c_1^{21}\alpha_2^{-1}(c_2)\otimes c_1^{22}=\sum\alpha_1^{-1}(c_1)(c_2^1)_1\otimes\alpha_2^{-1}(c_1)(c_2^1)_2\otimes c_2^2,$$

其中，$\forall c\in C$.

**定理2.2** 设 $C,H$ 都是弱 Hopf 代数，$S_C,S_H$ 分别为 $C$ 和 $H$ 的弱对极，若满足

$$\sum\varepsilon_C(c)\prod^l(h)=c_1^1\alpha_1(c_2)\prod^l(h)\cdot S_H(\alpha_1^{-1}(c_1^2)_2\alpha_2(c_2))\alpha_2^{-1}(c_1^2)_2,$$

则弱双代数 $C\times_\alpha H$ 为弱 Hopf 代数，并有弱对极

$$S_{C\times_\alpha H}(c\times h)=S_C(c)\times S_H(h).$$

**证明** 只需验证弱 Hopf 代数定义中的条件(4)，即

(1) $(c\times h)_1S_{C\times_\alpha H}((c\times h)_2)=\varepsilon_{C\times_\alpha H}((1_C\times 1_H)_1(c\times h))(1_C\times 1_H)_2$,

(2) $S_{C\times_\alpha H}((c\times h)_1)(c\times h)_2=(1_C\times 1_H)_1\varepsilon_{C\times_\alpha H}((c\times h)(1_C\times 1_H)_2)$,

(3) $S_{C\times_\alpha H}((c\times h)_1)(c\times h)_2S_{C\times_\alpha H}((c\times h)_3)=S_{C\times_\alpha H}(c\times h)$.

这里，只验证(1)，其余证明类似可得.

$$(c\times h)_1S_{C\times_\alpha H}((c\times h)_2)=\sum(c_1\times c_2^1\alpha_1(c_3)h_1)S_{C\times_\alpha H}(c_2^2\times\alpha_2(c_3)h_2)=\sum(c_1\times c_2^1\alpha_1(c_3)h_1)(S_C(c_2^2)\times S_H(\alpha_2(c_3)h_2))=\sum(c_1\times c_2^1\alpha_1(c_3)h_1)(S_C(c_2^2)_2\times S_H(\alpha_2(c_3)h_2)S_H(\alpha_1^{-1}(c_2^2)_1)\alpha_2^{-1}(c_2^2)_1)=\sum c_1S_C(c_2^2)_2\times c_2^1\alpha_1(c_3)h_1S_H(\alpha_2(c_3)h_2)S_H(\alpha_1^{-1}(c_2^2)_1)\alpha_2^{-1}(c_2^2)_1=\sum c_1S_C(c_2^2)_1\times c_2^1\alpha_1(c_3)h_1S_H(\alpha_2(c_3)h_2)S_H(\alpha_1^{-1}(c_2^2)_2)\alpha_2^{-1}(c_2^2)_2=\sum\prod^l(c_1)\times c_2^1\alpha_1(c_3)\prod^l(h)S_H(\alpha_1^{-1}(c_2^2)_2\alpha_2(c_3))\alpha_2^{-1}(c_2^2)_2=\prod^l(c_1)\times\prod^l(h),$$

$$\varepsilon_{C\times_\alpha H}((1_C\times 1_H)_1(c\times h))(1_C\times 1_H)_2=\sum\varepsilon_{C\times_\alpha H}((1_{C_1}\times 1_{C_2}^1\alpha_1(1_{C_3})1_{H_1})(c\times h))(1_{C_2}^2\times\alpha_2(1_{C_3})1_{H_2})=\sum\varepsilon_C(1_{C_1}c)1_{C_2}^2\times\varepsilon_H(1_{C_2}^1\alpha_1(1_{C_3})1_{H_1}h)\alpha_2(1_{C_3})1_{H_2}=\sum\varepsilon_C(1_{C_1}c)1_{C_2}^2\times\varepsilon_H(1_{C_2}^1)\varepsilon_H(\alpha_1(1_{C_3}))\varepsilon_H(1_{H_1}h)\alpha_2(1_{C_3})1_{H_2}=\sum\varepsilon_C(1_{C_1}c)1_{C_2}^2\times\varepsilon_H(1_{C_2}^1)\varepsilon_H(1_{H_1}h)\varepsilon_C(1_{C_3})1_{H_2}=\prod^l(c_1)\times\prod^l(h).$$

故 $C\times_\alpha H$ 是弱 Hopf 代数.

**推论2.1** 设 $C$ 是弱余代数，$H$ 是弱 Hopf 代数，则交叉余积 $C_\alpha[H]$ 为余结合余代数，当且仅当，对 $\forall c\in C$，

(1) $\sum\alpha_1(c_2)\otimes\alpha_1(c_1)(\alpha_2(c_2))_1\otimes\alpha_2(c_1)(\alpha_2(c_2))_2=\sum\alpha_1(c_1)(\alpha_1(c_2))_1\otimes\alpha_2(c_1)(\alpha_1(c_2))_2\otimes\alpha_2(c_2)$,

(2) $\sum\alpha_1(c_2)\otimes\alpha_2(c_2)\otimes c_1=\sum\alpha_1(c_1)\otimes\alpha_2(c_1)\otimes c_2$.

**证明** 由定义 1.2 与引理 1.2 可知，若 $\rho$ 是平凡的，即 $\rho(c)=c\otimes 1$，并记 $C\times_\alpha H$ 为 $C_\alpha[H]$. 同时，对 $\forall c\in C$，若满足 $\sum\varepsilon(\alpha_2(c))\alpha_1(c)=\varepsilon(c)1_H$，则 $C\times_\alpha H$ 是余结合余代数，当且仅当

(1) $\sum c_1^1\alpha_1(c_2)\otimes\alpha_1(c_1^2)(\alpha_2(c_2))_1\otimes\alpha_2(c_1^2)(\alpha_2(c_2))_2=\sum\alpha_1(c_1)(\alpha_1(c_2))_1\otimes\alpha_2(c_1)(\alpha_1(c_2))_2\otimes\alpha_2(c_2)$,

(2) $\sum c_1^1\alpha_1(c_2)\otimes c_1^{21}\alpha_2(c_2)\otimes c_1^{22}=\sum\alpha_1(c_1)(c_2^1)_1\otimes\alpha_2(c_1)(c_2^1)_2\otimes c_2^2$.

**推论 2.2** 设 $C$ 弱双代数，其他条件同推论 2.1，则 $C_\alpha[H]$ 是余结合余代数，且 $C\times_\alpha H\cong C_\alpha[H]$.

$C_\alpha[H]$ 是交叉余积的特例，即弱余作用是平凡的，故 $C_\alpha[H]$ 是弱 Hopf 代数.

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