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Finite Direct Sums of Some Rings*
某些环上的有限直和

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Abstract: Let S and T be rings, $R = S \oplus T$. We proved that R is a V-ring (resp., GV-ring; FS-ring; IF-ring; FC-ring; n-FC ring) if and only if both S and T are V-ring (resp., GV-ring; FS-ring; IF-ring; FC-ring; n-FC ring). In general, let R_1, R_2, $\cdots$, R_m be rings. We obtained that $\oplus_{i=1}^{m} R_i$ is a V-ring (resp., GV-ring; FS-ring; IF-ring; FC-ring; n-FC ring) if and only if each R_i is a V-ring (resp., GV-ring; FS-ring; IF-ring; FC-ring; n-FC ring).

Key words: V-ring, GV-ring, FS-ring, IF-ring, FC-ring, n-FC ring

摘要:证明 R 是 V-环(或 GV-环; FS-环; IF-环; FC-环; n-FC 环)当且仅当 S 和 T 都是 V-环(或 GV-环; FS-环; IF-环; FC-环; n-FC 环),其中 S 和 T 是环,$R = S \oplus T$. 一般地,当 $R = \oplus_{i=1}^{m} R_i$ 时,类似的结论也成立.

关键词: V-环 GV-环 FS-环 IF-环 FC-环 n-FC 环
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First we recall some known definitions which we needed in the sequel. Let R be a ring. Following reference[1], R is called right n-coherent in case every n-presented right R-module is ($n+1$)-presented. According to reference[2], R is called a right FS-ring in case its socle $Soc(R_R)$ is flat. R is said to be a right V-ring[3] if every simple right R-module is injective. R is called a right GV-ring[4] if every simple right R-module is injective or projective. Colby[5] called R a right IF-ring if every injective right R-module is flat. R is said to be a right FC-ring[6] if R is right coherent and FP-injective as a right R-module. Following reference[7], a ring R is called a right n-FC in case R is right n-coherent and R_R is n-FP-injective. A right R-module M is called n-FP-injective if $Ext_R^n(N,M) = 0$ for any n-presented right R-module N.

Let S and T be rings, and write $R = S \oplus T$. We proved that R is a V-ring (resp., GV-ring; FS-ring; IF-ring; FC-ring; n-FC ring) if and only if both S and T are V-ring (resp., GV-ring; FS-ring; IF-ring; FC-ring; n-FC ring). In general, let R_1, R_2, $\cdots$, R_m be rings. We obtain that $\oplus_{i=1}^{m} R_i$ is a V-ring (resp., GV-ring; FS-ring; IF-ring; FC-ring; n-FC ring) if and only if each R_i is a V-ring (resp., GV-ring; FS-ring; IF-ring; FC-ring; n-FC ring).

Throughout this paper all rings are associative with identity and modules are unitary. n and d are non-negative integers. Definitions and notations not given here can be found in reference[8,9].

Lemma 1[10] Let R and S be rings. Every right ($R \oplus S$)-module has a unique decomposition that $M = A \oplus B$, where $A = M(R,0)$ is a right R-module and $B = M(0,S)$ is a right S-module via $xr = x(r,0)$ for $x \in A$, $r \in R$, and $ys = y(0,s)$ for $y \in B$, $s \in S$.

Lemma 2[11] Let $f: R \to S$ be a surjective ring homomorphism. If M_S is a right S-module (hence a right R-module) and A_R is a right R-module, then M

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$\oplus_R S_S \cong M_S$.

Let R be a ring. Recall that a non-zero right R-module M is called simple[8] if it has non-trivial submodules.

Lemma 3 Let R be a ring and S be a summand of R . If M_S is a right S -module, then the following statements hold:

(1) M_S is a simple right S -module if and only if M_R is a simple right R -module.

(2) M_S is a projective right S -module if and only if M_R is a projective right R -module.

Proof (1) Since S is a summand of R , there exists a ring T such that $R = S \oplus T$. Note that M is a right R -module. By Lemma 1, M has a unique decomposition that $M = A \oplus B$, where A is a right S-module and B is a right T -module. By the uniqueness of the decomposition, we have $A = M$ and $B = 0$. So (1) follows.

(2) If M_S is projective, then M_R is projective by the Theorem 11. 69 of reference[9]. Conversely, If M_R is projective, then $M \oplus_R S_S$ is a projective right S -module by the Theorem11. 60 of reference [9]. But $M \oplus_R S_S \cong M_S$ by Lemma 2. So M_S is projective.

Lemma 4 Let $f: R \to S$ be a surjective ring homomorphism, M_S a right S -module and ${}_SA$ a left S-module. If both S_R and ${}_RS$ are projective, then the following statements hold:

(1) M_S is an injective right S -module if and only if M_R is an injective right R -module.

(2) M_S is a FP -injective right S -module if and only if M_R is a FP -injective right R -module.

(3) ${}_SA$ is a flat left S -module if and only if ${}_RA$ is a flat left R -module.

(4) If R is a right n -coherent ring, then S is a right n-coherent ring.

(5) M_S is an n - FP -injective right S -module if and only if M_R is an n-FP-injective right R -module.

Proof (1)(2)(3)(4). Following reference[1], M is a (0,0)-injective (resp. (1,0)-injective, (1,0)-flat) if and only if M is injective (resp. FP -injective, flat). The proof are clear by reference[11].

(5)⇒. Suppose M_S is an n - FP -injective right S -module. Let N_R be any n -presented right R -module. Then, using an argument similar to that Lemma 2 of reference[11], we get that $N \otimes_R S_S$ is an n-presented right S -module. By the Theorem 11. 65 of reference[9], we have

$$Ext_R^n(N_R, M_R) \cong Ext_S^n(N \otimes_R S_S, M_S) = 0.$$

Therefore M_R is an n -FP -injective right R -module.

⇐. Assume M_R is an n - FP -injective right R-module, N_S be any n -presented right S -module. Then $N \otimes_R S_S \cong N_S$ by Lemma 2, and N_R is an n-presentedright R -module by the Lemma 2 of reference [11]. Again by the Theorem 11. 65 of reference[9], we have

$$Ext_S^n(N_S, M_S) \cong Ext_S^n(N \otimes_R S_S, M_S) \cong Ext_R^n(N_R, M_R) = 0.$$

Therefore M_S is an n - FP -injective right S -module.

Theorem 1 Let S and T be rings, and write $R = T \oplus S$. Then the following are true:

(1) R is a right V-ring if and only if both S and T are right V-rings.

(2) R is a right GV -ring if and only if both S and T are right GV-rings.

(3) R is a right FS -ring if and only if both S and T are right FS-rings.

(4) R is a right IF-ring if and only if both S and T are right IF-rings.

(5) R is a right FC -ring if and only if both S and T are right FC-rings.

(6) R is a right n - FC ring if and only if both S and T are right n -FC rings.

Proof (1)⇒. Let M_S be a simple right S -module, then M_R is a simple right R -module by Lemma 3. So M_R is an injective right R -module since R is a right V-ring. Thus M_S is an injective right S -module by Lemma 4. Therefore, S is a right V-ring. Similarly, we get that T is a right V-ring.

⇐ . Let M_R be a simple right R -module. Then $M = A \oplus B$, where A is a right S -module and B is a right T -module by Lemma 1. Since M_R is simple, we have either $A = 0$ or $B = 0$. We may assume with out loss of generality that $A = 0$. Then $B \cong M$ as groups, and hence M is a right T -module since B is a right T -module. It follows that M is a simple right T -module by Lemma 3. Hence M is an injective right T -module since T is a right V-ring. Whence M is an injective right R -module by Lemma 4. Therefore R is a right V-ring.

(2) Similar to the proof of (1).

(3) Note that $Soc(R_R) = Soc(S_S) \oplus Soc(T_T)$ by the Proposition 9.19 of reference[8]. So it is easy to see that $Soc(R_R)$ is a flat right R-module if and only if $Soc(S_S)$ is a flat right S-module and $Soc(T_T)$ is a flat right T-module by Lemma 4. Hence (3) follows.

(4) $\Rightarrow$. Let M_S be an injective right S-module. Then M_R is an injective right R-module by Lemma 4. So M_R is a flat right R-module since R is a right IF-ring. Thus M_S is a flat right S-module again by Lemma 4. Therefore, S is a right IF-ring. Similarly, we get that T is a right IF-ring.

$\Leftarrow$. Let M_R be an injective right R-module. Then $M = A \oplus B$, where A is a right S-module and B is a right T-module by Lemma 1. Note that both A and B are injective right R-modules (for M_R is an injective right R-module). Thus A is an injective right S-module and B is an injective right T-module by Lemma 4. Since both S and T are right IF-rings, we have A is a flat right S-module and B is a flat right T-module. Whence both A and B are flat right R-modules again by Lemma 4, and so $M = A \oplus B$ is a flat right R-module. Therefore, R is a right IF-ring.

(5) By the Theorem 1 of reference[11], R is a right coherent ring if and only if both S and T are right coherent rings. By Lemma 4, R is a right FP-injective R-module if and only if S is a right FP-injective S-module and T is a right FP-injective T-module. Thus (5) follows.

(6) Similar to the proof of (5).

Corollary 1 Let $R_1, R_2, \cdots, R_m$ be rings. Then the following statements hold:

(1) $\bigoplus_{i=1}^{m} R_i$ is a right V-ring if and only if each R_i is a right V-ring.

(2) $\bigoplus_{i=1}^{m} R_i$ is a right GV-ring if and only if each R_i is a right GV-ring.

(3) $\bigoplus_{i=1}^{m} R_i$ is a right FS-ring if and only if each R_i is a right FS-ring.

(4) $\bigoplus_{i=1}^{m} R_i$ is a right IF-ring if and only if each R_i is a right IF-ring.

(5) $\bigoplus_{i=1}^{m} R_i$ is a right FC-ring if and only if each R_i is a right FC-ring.

(6) $\bigoplus_{i=1}^{m} R_i$ is a right n-FC ring if and only if each R_i is a right n-FC ring.

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