Some Subclasses of Analytic Functions Associated with a Linear Operator *

几类与线性算子相关的解析函数族

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Abstract: Some new subclasses of analytic functions associated with a certain linear operator were investigated. Some properties were proved such as subordination, superordination, integral-preserving, convolution, inclusion relationships. Several sandwich-type results were also derived.

Key words: analytic functions, hadamard product (or convolution), subordination, super-subordination, linear operator

摘要:研究几类和线性算子相关的解析函数族,给出它们的从属与超从属性质,积分保持性质,卷积,包含关系,并得到一些"sandwich"型双从属结果.

关键词:解析函数　卷积　从属　超从属　线性算子

中图法分类号:O174.51　　**文献标识码**:A　　**文章编号**:1005-9164(2010)03-0188-06

Let A denote the class of functions of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k, \qquad (0.1)$$

which are analytic in the open unit disk

$$U:=\{z:z\in C \text{ and } |z|<1\}.$$

Let $H(U)$ be the linear space of all analytic functions in U. For a positive integer number n and $a\in C$, we let

$$H[a,n]:=\{f\in H(U):f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\cdots\}.$$

Let P denote the class of functions of the form:

$$p(z)=1+\sum_{k=1}^{\infty}p_kz^k \quad (n\in N),$$

which are analytic and convex in U and satisfy the condition: $R(p(z))>0$ $(z\in U)$. Let $f,g\in A$, where f is given by form (0.1) and g is defined by $g(z)=z+\sum_{k=2}^{\infty}b_kz^k$. Then the Hadamard product (or convolution) $f*g$ of the functions f and g is defined by

$$(f*g)(z):=z+\sum_{k=2}^{\infty}a_kb_kz^k=:(g*f)(z).$$

For two functions f and g, analytic in U, we say that the function f is subordinate to g in U, and write

$$f(z)\prec g(z) \quad (z\in U),$$

if there is a Schwarz function ω, which is analytic in U with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in U)$, such that $f(z)=g((\omega(z))$ $(z\in U)$.

Indeed, it is known that

$$f(z)\prec g(z)(z\in U)\Rightarrow f(0)=g(0) \text{ and } f(U)\subset g(U).$$

Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z)\prec g(z)(z\in U)\Leftrightarrow f(0)=g(0) \text{ and } f(U)\subset g(U).$$

For complex parameters

$\alpha_j\in C$ $(j=1,2,\cdots,q)$ and $\beta_j\in C\backslash Z_0^-$

收稿日期:2010-02-24

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* The present investigation was supported by the Scientific Research Fund of Guangdong Provincial Education Department under Grant LYM08101 and the Scientific Research Fund of Hunan Provincial Education Department under Grant 08C118 of People's Republic of China.

$(Z_0^- := \{0, -1, -2, \cdots\}; j = 1, 2, \cdots, s)$,
the generalized hypergeometric function ${}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$ is defined by the infinite series:

$$ {}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \cdot \frac{z^n}{n!} \quad (q \leqq s+1; q, s \in N_0 := N \cup \{0\}; N := \{1, 2, 3, \cdots\}; z \in U), $$

where $(\mu)_n$ is the Pochhammer symbol defined by

$$ (\mu)_n := \begin{cases} 1 (n = 0), \\ \mu(\mu+1)\cdots(\mu+n-1) \quad (n \in N). \end{cases} $$

Corresponding to the function $h(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$, defined by

$$ h(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) := z\,{}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z). \qquad (0.2) $$

A linear operator is considered in reference[1~7]:

$$ H(\alpha_1, \cdots, \alpha_q; \beta, \cdots, \beta_s): A \to A, $$

defined by the following Hadamard product:

$$ H(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s) f(z) := h(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) * f(z) \quad (q \leqq s+1; q, s \in N_0; z \in U; f \in A). $$

We note that the linear operator $H(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s)$ includes various other linear operators which were introduced and studied earlier in reference[8~12].

Corresponding to the function $h(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$ defined by form(0.2), we introduce a function $h_\lambda(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$ given by

$$ h(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) * h_\lambda(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = \frac{z}{(1-z)^\lambda} \ (\lambda > 0). \qquad (0.3) $$

Analogous to the Dziok-Srivastava operator $H(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s)$, a new linear operator $H_\lambda(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s)$ was definedas in references[13,14] as follows:

$$ H_\lambda(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s) f(z) := h_\lambda(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) * f(z). \qquad (0.4) $$

For convenience, we write

$$ H_\lambda^{q,s}(\alpha_j) := H_\lambda(\alpha_1, \cdots, \alpha_j, \cdots, \alpha_q; \beta_1, \cdots, \beta_s) \quad (j \in \{1, 2, \cdots, q\}). \qquad (0.5) $$

According to the forms(0.2), (0.3) and (0.4), we can easily find that

$$ H_\lambda^{q,s}(\alpha_j) f(z) = z + \sum_{k=2}^{\infty} \frac{(\lambda)_{k-1} (\beta_1)_{k-1} \cdots (\beta_s)_{k-1}}{(\alpha_1)_{k-1} \cdots (\alpha_j)_{k-1} \cdots (\alpha_q)_{k-1}} \cdot a_k z^k \quad (z \in U). \qquad (0.6) $$

We note that $H_\lambda^{2,1}(\gamma+1, 1; 1)$ $(\lambda > 0; \gamma > -1)$ is the well-known Choi-Saigo-Srivastava operator[15~17].

It is readily verified from the definition (0.4) that

$$ z(H_\lambda^{q,s}(\alpha_j+1) f)'(z) = \alpha_j H_\lambda^{q,s}(\alpha_j) f(z) - (\alpha_j - 1) H_\lambda^{q,s}(\alpha_j+1) f(z), \qquad (0.7) $$

$$ z(H_\lambda^{q,s}(\alpha_j) f)'(z) = \lambda H_{\lambda+1}^{q,s}(\alpha_j) f(z) - (\lambda - 1) H_\lambda^{q,s}(\alpha_j) f(z). \qquad (0.8) $$

By using the subordination between analytic functions and the operator $H_\lambda^{q,s}(\alpha_j)$, we introduce subclasses of analytic functions. And in the present paper, we aim at proving some subordination and superordination properties, inclusion relationships, integral-preserving properties and convolution properties associated with the operator $H_\lambda^{q,s}(\alpha_j)$. Several sandwich-type results involving this operator are also derived.

1 Definitions and preliminary results

Definition 1 A function $f \in A$ is said to be in the class $F_\lambda^{q,s}(\alpha_j; \delta; \phi)$ if it satisfies the following subordination condition:

$$ (1-\delta) \frac{H_\lambda^{q,s}(\alpha_j+1) f(z)}{z} + \delta \frac{H_\lambda^{q,s}(\alpha_j) f(z)}{z} \prec \phi(z) \quad (z \in U; \delta \in C; \phi \in P). \qquad (1.1) $$

Definition 2 A function $f \in A$ is said to be in the class $G_\lambda^{q,s}(\alpha_j; \delta; \phi)$ if it satisfies the following subordination condition:

$$ (1-\delta) \frac{H_\lambda^{q,s}(\alpha_j) f(z)}{z} + \delta \frac{H_{\lambda+1}^{q,s}(\alpha_j) f(z)}{z} \prec \phi(z) \quad (z \in U; \delta \in C; \phi \in P). \qquad (1.2) $$

Lemma 1[18] Let $\theta, \gamma \in C$. Suppose that φ is convex and univalent in U with $\varphi(0) = 1$ and $R(\theta\varphi(z) + \gamma) > 0$ $(z \in U)$. If p is analytic in U with $p(0) = 1$, then the following subordination:

$$ p(z) + \frac{z p'(z)}{\theta p(z) + \gamma} \prec \varphi(z) \ (z \in U) $$ implies that $p(z) \prec \varphi(z)$ $(z \in U)$.

Lemma 2[19] Let the function Ω be analytic and convex (univalent) in U with $\Omega(0) = 1$. Suppose also that the function Θ given by

$$ \Theta(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots $$

is analytic in U. If

$$ \Theta(z) + \frac{z\Theta'(z)}{\zeta} \prec \Omega(z) \quad (R(\zeta) > 0; \zeta \neq 0; z \in U), \qquad (1.3) $$

then

$$ \Theta(z) \prec \chi(z) = \frac{\zeta}{n} z^{-\frac{\zeta}{n}} \int_0^z t^{\frac{\zeta}{n}-1} h(t) \mathrm{d}t \prec \Omega(z) \quad (z \in U), $$

and χ is the best dominant of form(1.3).

Lemma 3[20] Let F be analytic andconvex in U. If $f, g \in A$ and $f, g \prec F$, then $\gamma f + (1-\gamma) g \prec F$ $(0 \leqq \gamma \leqq 1)$.

Let Q denote the set of all functions f that are analytic and injective on $\overline{U} - E(f)$, where $E(f) = \{\varepsilon \in \partial U : \lim_{z \to \varepsilon} f(z) = \infty\}$, and such that $f'(\varepsilon) \neq 0$ for $\varepsilon \in \partial U - E(f)$.

Lemma 4[21] Let q be convex univalent in U and $\kappa \in C$. Further assume that $R(\bar{\kappa}) > 0$. If $p(z) \in H[q(0),1] \cap Q$, and $p(z) + \kappa z p'(z)$ is univalent in U, then $q(z) + \kappa z q'(z) \prec p(z) + \kappa z p'(z)$ implies $q(z) \prec p(z)$ and q is the best subdominant.

Lemma 5[22] Let q be a convex univalent function in U and let $\sigma, \eta \in C$ with

$$R\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\{0, -R(\frac{\sigma}{\eta})\}.$$

If p is analytic in U and $\sigma p(z) + \eta z p'(z) \prec \sigma q(z) + \eta z q'(z)$, then $p(z) \prec q(z)$, and q is the best dominant.

Lemma 6[23] Let the function Υ be analytic in U with

$$\Upsilon(0) = 1 \text{ and } R(\Upsilon(z)) > \frac{1}{2} \quad (z \in U).$$

Then, for any function Ψ analytic in U, $(\Upsilon * \Psi)(U)$ is contained in the convex hull of $\Psi(U)$.

2 Properties of the function class $F_\lambda^{q,s}(\alpha_j;\delta;\phi)$

Theorem 1 Let $f \in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$ with $R(\frac{\delta}{\alpha_j}) > 0$. Then

$$\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z} \prec \frac{\alpha_j}{\delta} z^{-\frac{\alpha_j}{\delta}} \int_0^z t^{\frac{\alpha_j}{\delta}-1}\phi(t)\mathrm{d}t \prec \phi(z)$$

$(z \in U)$. (2.1)

Proof Let $f \in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$ and suppose that

$$h(z) := \frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z} \quad (z \in U). \qquad (2.2)$$

Then h is analytic in U. By virtue of forms (0.7), (1.1) and (2.2), we find that

$$h(z) + \frac{\delta}{\alpha_j} z h'(z) = (1-\delta)\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z} + \delta \frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z} \prec \phi(z) \quad (z \in U). \qquad (2.3)$$

Thus, an application of Lemma 2 for $n = 1$ to form (2.3) yields the assertion (2.1) of Theorem 1.

In view of Theorem 1, we easily get the following inclusion relationship.

Corollary 1 Let $R(\frac{\delta}{\alpha_j}) > 0$. Then

$F_\lambda^{q,s}(\alpha_j;\delta;\phi) \subset F_\lambda^{q,s}(\alpha_j;0;\phi)$.

Next, we prove another inclusion relationship for the function class $F_\lambda^{q,s}(\alpha_j;\delta;\phi)$.

Theorem 2 Let $\delta_2 > \delta_1 \geqq 0$. Then

$F_\lambda^{q,s}(\alpha_j;\delta_2;\phi) \subset F_\lambda^{q,s}(\alpha_j;\delta_1;\phi)$.

Proof Suppose that $f \in F_\lambda^{q,s}(\alpha_j;\delta_2;\phi)$. It follows that

$$(1-\delta_2)\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z} + \delta_2 \frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z} \prec \phi(z) \quad (z \in U). \qquad (2.4)$$

Since $0 \leqq \frac{\delta_1}{\delta_2} < 1$, and the function ϕ is convex and univalent in U, from forms (2.1), (2.4) and Lemma 3 we deduce that

$$(1-\delta_1)\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z} + \delta_1 \frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z} = \frac{\delta_1}{\delta_2}\left[(1-\delta_2)\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z} + \delta_2\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}\right] + \left(1-\frac{\delta_1}{\delta_2}\right)\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z} \prec \phi(z) \quad (z \in U),$$

which implies that $f \in F_\lambda^{q,s}(\alpha_j;\delta_1;\phi)$. The proof of Theorem 2 is evidently completed.

Theorem 3 Let $f \in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$. If the integral operator F is defined by

$$F(z) := \frac{\upsilon+1}{z^\upsilon}\int_0^z t^{\upsilon-1} f(t)\mathrm{d}t \quad (z \in U; \upsilon > -1), \qquad (2.5)$$

then

$$\frac{H_\lambda^{q,s}(\alpha_j+1)F(z)}{z} \prec \phi(z) \quad (z \in U). \qquad (2.6)$$

Proof Let $f \in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$. Suppose also that

$$G(z) := \frac{H_\lambda^{q,s}(\alpha_j+1)F(z)}{z} \quad (z \in U). \qquad (2.7)$$

From form (2.5), we deduce that

$$z(H_\lambda^{q,s}(\alpha_j+1)F)'(z) + \upsilon H_\lambda^{q,s}(\alpha_j+1)F(z) = (\upsilon+1)H_\lambda^{q,s}(\alpha_j+1)f(z). \qquad (2.8)$$

Combining forms (2.1), (2.7) and (2.8), we have

$$G(z) + \frac{1}{\upsilon+1} z G'(z) = \frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z} \prec \phi(z) \quad (z \in U). \qquad (2.9)$$

Thus, by Lemma 2 and form (2.9), we conclude that the assertion (2.6) of Theorem 3 holds true.

Theorem 4 Let $f \in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$ and $g \in A$ with $R(\frac{g(z)}{z}) > \frac{1}{2}$. Then $(f * g)(z) \in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$.

Proof Let $f \in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$ and $g \in A$ with $R(\frac{g(z)}{z}) > \frac{1}{2}$. Suppose also that

$$K(z) := (1-\delta)\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z} + \delta\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z} \prec \phi(z) \quad (z \in U). \qquad (2.10)$$

It follows form (2.10) that

$$(1-\delta)\frac{H_\lambda^{q,s}(\alpha_j+1)(f*g)(z)}{z} + \delta\frac{H_\lambda^{q,s}(\alpha_j)(f*g)(z)}{z} = K(z) * \frac{g(z)}{z} \quad (z \in U). \qquad (2.11)$$

Since the function ϕ is convex and univalent in U, by means of forms(2.10),(2.11) and Lemma 6, we conclude that

$$(1-\delta)\frac{H_\lambda^{q,s}(\alpha_j+1)(f*g)(z)}{z}+\delta\frac{H_\lambda^{q,s}(\alpha_j)(f*g)(z)}{z}\prec\phi(z)\ (z\in U),\qquad(2.12)$$

which implies that the assertion of Theorem 4 holds true.

Theorem 5 Let $f\in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$ and F is defined by form(2.5) with $f\in A$ and $\upsilon>-1$. Then $F\in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$.

Proof Let $f\in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$ and F is defined by form(2.5) with $\upsilon>-1$. We easily find that

$$F(z)=\frac{\upsilon+1}{z^\upsilon}\int_0^z t^{\upsilon-1}f(t)\mathrm{d}t=(f*h)(z),$$

where

$$h(z)=\frac{\upsilon+1}{z^\upsilon}\int_0^z\frac{t^\upsilon}{1-t}\mathrm{d}t\in A.$$

Moreover, for $\upsilon>-1$, we have

$$R(\frac{h(z)}{z})=R(\frac{\upsilon+1}{z^{\upsilon+1}}\int_0^z\frac{t^\upsilon}{1-t}\mathrm{d}t)=(\upsilon+1)\int_0^1u^\upsilon R(\frac{1}{1-uz})\mathrm{d}u>(\upsilon+1)\int_0^1\frac{u^\upsilon}{1+u}\mathrm{d}u>\frac{1}{2}$$

$$(z\in U).\qquad(2.13)$$

Combining form(2.13) and Theorem 4, we conclude that $F\in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$. The proof of Theorem 5 is thus completed.

Theorem 6 Let $f\in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$ and

$$S_m(z):=z+\sum_{k=2}^m a_kz^k\ (z\in U;m\in N\backslash\{1\}).\qquad(2.14)$$

Then the function W_m defined by

$$W_m(z):=\int_0^z\frac{S_m(t)}{t}\mathrm{d}t\ (z\in U;m\in N\backslash\{1\})$$

belongs to the class $F_\lambda^{q,s}(\alpha_j;\delta;\phi)$.

Proof Let $f\in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$ and S_m be defined by form (2.14). We readily get

$$W_m(z)=\int_0^z\frac{S_m(t)}{t}\mathrm{d}t=(f*g_m)(z)\ (z\in U;m\in N\backslash\{1\}),$$

where

$$g_m(z)=z+\sum_{k=2}^m\frac{z^k}{k}\in A\ (z\in U).$$

For $m\in N\backslash\{1\}$, we know from reference[20] that

$$R(\frac{g_m(z)}{z})=R(1+\sum_{k=2}^m\frac{z^{k-1}}{k})>\frac{1}{2}\ (z\in U).\qquad(2.15)$$

Combining form(2.15) and Theorem 4, we deduce that $W_m\in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$. We thus complete the proof of Theorem 6.

Theorem 7 Let $f\in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$. Then

$$\frac{1}{z}[(z+\sum_{k=2}^\infty\frac{(\lambda)_{k-1}(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}}{(\alpha_1)_{k-1}\cdots(\alpha_j+1)_{k-1}\cdots(\alpha_q)_{k-1}}z^k)*f(z)-z\phi(e^{i\theta})]\neq0\ (z\in U;0\leqq\theta<2\pi).\qquad(2.16)$$

Proof Suppose that $f\in F_\lambda^{q,s}(\alpha_j;\delta;\phi)$. We know that form (2.1) holds, which can be rewritten as follows:

$$\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z}\neq\phi(e^{i\theta})\ (z\in U;0\leqq\theta<2\pi).\qquad(2.17)$$

Furthermore, from form(0.6) we find that

$$H_\lambda^{q,s}(\alpha_j+1)f(z)=(z+\sum_{k=2}^\infty\frac{(\lambda)_{k-1}(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}}{(\alpha_1)_{k-1}\cdots(\alpha_j+1)_{k-1}\cdots(\alpha_q)_{k-1}}z^k)*f(z)\ (z\in U).\qquad(2.18)$$

Now the assertion (2.16) of Theorem 7 can be easily derived from forms(2.17) and (2.18).

Theorem 8 Let q_1 be univalent in U and $R(\frac{\delta}{\alpha_j})>0$. Suppose also that q_1 satisfies

$$R(1+\frac{zq''_1(z)}{q'_1(z)})>\max\{0,-R(\frac{\alpha_j}{\delta})\}.\qquad(2.19)$$

If $f\in A$ satisfies the following subordination:

$$(1-\delta)\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z}+\delta\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}\prec q_1(z)+\frac{\delta}{\alpha_j}zq_1'(z),\qquad(2.20)$$

then

$$\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z}\prec q_1(z),$$ and q_1 is the best dominant.

Proof Let the function h be defined by form (2.2). We know that form (2.3) holds true. Combining forms(2.3) and (2.20), we find that

$$h(z)+\frac{\delta}{\alpha_j}zh'(z)\prec q_1(z)+\frac{\delta}{\alpha_j}zq_1'(z).\qquad(2.21)$$

By Lemma 5 and form (2.21), we easily get the assertion of Theorem 8.

If f is subordinate to F, then F is superordinate to f. We now derive the following superordination result.

Theorem 9 Let q_2 be convex univalent in U, $\delta\in C$ with $R(\frac{\delta}{\alpha_j})>0$. Also let

$$\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z}\in H[q_2(0),1]\cap Q,$$

and

$$(1-\delta)\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z}+\delta\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}$$

be univalent in U. If $q_2(z)+\frac{\delta}{\alpha_j}zq_2'(z)\prec(1-\delta)\cdot\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z}+\delta\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}$, then $q_2(z)\prec$

$\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z}$, and q_2 is the best subordinant.

Proof Let the function h be defined by form (2.2). Then

$$q_2(z)+\frac{\delta}{\alpha_j}zq_2'(z)\prec(1-\delta)[H_\lambda^{q,s}(\alpha_j+1)f(z)]/z+\delta\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}=h(z)+\frac{\delta}{\alpha_j}zh'(z).$$

An application of Lemma 4 yields the assertion of Theorem 9.

Combining the above results of subordination and superordination, we easily get the following "sandwich-type result".

Theorem 10 Let q_3 be convex univalent and let q_4 be univalent in U, $\delta\in C$ with $R(\frac{\delta}{\alpha_j})>0$. Suppose also that q_4 satisfies

$$R(1+\frac{zq''_4(z)}{q'_4(z)})>\max\{0,-R(\frac{\alpha_j}{\delta})\}.$$

If

$$0\neq\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z}\in H[q_3(0),1]\cap Q,$$

and

$$(1-\delta)\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z}+\delta\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}$$

is univalent in U, also

$$q_3(z)+\frac{\delta}{\alpha_j}zq'_3(z)\prec(1-\delta)[H_\lambda^{q,s}(\alpha_j+1)f(z)]/z+\delta\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}\prec q_4(z)+\frac{\delta}{\alpha_j}zq'_4(z),$$

then

$$q_3(z)\prec\frac{H_\lambda^{q,s}(\alpha_j+1)f(z)}{z}\prec q_4(z),$$

and q_3 and q_4 are the best subordinant and the best dominant, respectively.

3 Properties of the function class $G_\lambda^{q,s}(\alpha_j;\delta;\phi)$

In view of form (0.8), and by similarly applying the methods of proof of Theorems 1~10, respectively, we easily get the following properties for the function class $G_\lambda^{q,s}(\alpha_j;\delta;\phi)$. Here we choose some properties but omit the details involved.

Corollary 2 Let $f\in G_\lambda^{q,s}(\alpha_j;\delta;\phi)$ with $R(\frac{\delta}{\lambda})>0$. Then

$$\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}\prec\frac{\lambda}{\delta},z^{-\frac{\lambda}{\delta}}\int_0^z t^{\frac{\lambda}{\delta}-1}\phi(t)dt\prec\phi(z)$$

$(z\in U)$.

Corollary 3 Let $\delta_2>\delta_1\geqq 0$. Then
$G_\lambda^{q,s}(\alpha_j;\delta_2;\phi)\subset G_\lambda^{q,s}(\alpha_j;\delta_1;\phi)$.

Corollary 4 Let $f\in G_\lambda^{q,s}(\alpha_j;\delta;\phi)$. If the integral operator F is defined by form (2.5), then

$$\frac{H_\lambda^{q,s}(\alpha_j)F(z)}{z}\prec\phi(z)\ (z\in U).$$

Corollary 5 Let $f\in G_\lambda^{q,s}(\alpha_j;\delta;\phi)$ and $g\in A$ with $R(\frac{g(z)}{z})>\frac{1}{2}$. Then $(f*g)(z)\in G_\lambda^{q,s}(\alpha_j;\delta;\phi)$.

Corollary 6 Let $f\in G_\lambda^{q,s}(\alpha_j;\delta;\phi)$ and F is defined by form (2.5) with $f\in A$ and $\upsilon>-1$. Then $F\in G_\lambda^{q,s}(\alpha_j;\delta;\phi)$.

Corollary 7 Let $f\in G_\lambda^{q,s}(\alpha_j;\delta;\phi)$ and

$$S_m(z):=z+\sum_{k=2}^{m}a_kz^k\quad(z\in U;m\in N\backslash\{1\}).$$

Then the function W_m defined by

$$W_m(z):=\int_0^z\frac{S_m(t)}{t}dt\ (z\in U;m\in N\backslash\{1\}),$$

belongs to the class $G_\lambda^{q,s}(\alpha_j;\delta;\phi)$.

Corollary 8 Let $f\in G_\lambda^{q,s}(\alpha_j;\delta;\phi)$. Then

$$\frac{1}{z}[(z+\sum_{k=2}^{\infty}\frac{(\lambda)_{k-1}(\beta_1)_{k-1}\cdots(\beta_s)_{k-1}}{(\alpha_1)_{k-1}\cdots(\alpha_j)_{k-1}\cdots(\alpha_q)_{k-1}}z^k)*f(z)-z\phi(e^{i\theta})]\neq 0\ (z\in U;0\leqq\theta<2\pi).$$

Corollary 9 Let q_5 be univalent in U and $R(\frac{\delta}{\lambda})>0$. Suppose also that q_5 satisfies

$$R(1+\frac{zq''_5(z)}{q'_5(z)})>\max\{0,-R(\frac{\lambda}{\delta})\}.$$

If $f\in A$ satisfies the following subordination:

$$(1-\delta)\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}+\delta\frac{H_{\lambda+1}^{q,s}(\alpha_j)f(z)}{z}\prec q_5(z)+\frac{\delta}{\lambda}zq_5'(z),$$

then $\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}\prec q_5(z)$, and q_5 is the best dominant.

Corollary 10 Let q_6 be convex univalent in U, $\delta\in C$ with $R(\frac{\delta}{\lambda})>0$. Also let

$$\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}\in H[q_6(0),1]\cap Q,$$

and

$$(1-\delta)\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}+\delta\frac{H_{\lambda+1}^{q,s}(\alpha_j)f(z)}{z}$$

be univalent in U. If

$$q_6(z)+\frac{\delta}{\lambda}zq'_6(z)\prec(1-\delta)\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}+\delta\frac{H_{\lambda+1}^{q,s}(\alpha_j)f(z)}{z},$$

then $q_6(z)\prec\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z}$, and q_6 is the best subordinant.

Corollary 11 Let q_7 be convex univalent and let q_8 be univalentin U, $\delta\in C$ with $R(\frac{\delta}{\lambda})>0$. Suppose also that q_8 satisfies

$$R(1+\frac{zq''_8(z)}{q'_8(z)})>\max\{0,-R(\frac{\lambda}{\delta})\}.$$

If

$$0 \neq \frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z} \in H[q_7(0),1] \cap Q,$$

and

$$(1-\delta)\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z} + \delta\frac{H_{\lambda+1}^{q,s}(\alpha_j)f(z)}{z}$$

is univalent in U, also

$$q_7(z) + \frac{\delta}{\lambda}zq'_7(z) \prec (1-\delta)\frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z} + \delta\frac{H_{\lambda+1}^{q,s}(\alpha_j)f(z)}{z} \prec q_8(z) + \frac{\delta}{\lambda}zq'_8(z),$$

then

$$q_7(z) \prec \frac{H_\lambda^{q,s}(\alpha_j)f(z)}{z} \prec q_8(z),$$

and q_7 and q_8 are the best subordinant and the best dominant, respectively.

References:

[1] Liu J L, Srivastava H M. A class of multivalently analytic functions associated with the Dziok-Srivastava operator[J]. Integral Transforms Spec Funct, 2009, 20: 401-417.

[2] Srivastava H M, Yang D G, Xu N. Subordinations for multivalent analytic functions associated with the Dziok-Srivastava operator[J]. Integral Transforms Spec Funct, 2009, 20: 581-606.

[3] Wang Z G, Jiang Y P, Srivastava H M. Some subclasses of multivalent analytic functions involving the Dziok-Srivastava operator[J]. Integral Transforms Spec Funct, 2008, 19: 129-146.

[4] Dziok J, Srivastava H M. Classes of analytic functions associated with the generalized hypergeometric function [J]. Appl Math Comput, 1999, 103: 1-13.

[5] Dziok J, Srivastava H M. Some subclasses of analytic functions with fixed argument of coefficients associated with the generalized hypergeometric function [J]. Adv Stud Contemp Math, 2002(5): 115-125.

[6] Dziok J, Srivastava H M. Certain subclasses of analytic functions associated with the generalized linebreak hypergeometric function [J]. Integral Transforms Spec Funct, 2003, 14: 7-18.

[7] Huang Y Y, Liu M S. Properties of certain subclasses of multivalent analytic functions involving the Dziok-Srivastava operator [J]. Appl Math Comput, 2008, 204: 137-149.

[8] Hohlov Y E. Operators and operations in the class of univalent functions [J]. Izvestiya Vysshikh Uchebnykh Zavedenii Matematika, 1978, 10: 83-89.

[9] Carlson B C, Shaffer D B. Starlike and prestarlike hypergeometic functions[J]. SIAM J Math Anal, 1984, 15: 737-745.

[10] Goel R M, Sohi N S. A new criterion for p-valent functions[J]. Proc Amer Math Soc, 1980, 78: 353-357.

[11] Owa S, Srivastava H M. Some applications of the generalized Libera integral operator[J]. Proc Japan Acad Ser A, 1986, 62: 125-128.

[12] Ruscheweyh S. New criteria for univalent functions[J]. Proc Amer Math Soc, 1975, 49: 109-115.

[13] Al-Kharsani H A, Al-Areefi N M. On classes of multivalent functions involving linear operator and multiplier transformations[J]. Hacet J Math Stat, 2008, 37: 115-127.

[14] Kwon O S, Cho N E. Inclusion properties for certain subclasses of analytic functions associated with the Dziok-Srivastava operator [J]. J Inequal Appl, 2007, Article ID 51079: 1-10.

[15] Ling Y, Liu F S. The Choi-Saigo-Srivastava integral operator and a class of analytic functions[J]. Appl Math Comput, 2005, 165: 613-621.

[16] Sókol J. Classes of analytic functions associated with the Choi-Saigo-Srivastava operator[J]. J Math Anal Appl, 2006, 318: 517-525.

[17] Choi J H, Saigo M, Srivastava H M. Some inclusion properties of a certain family of integral operators[J]. J Math Anal Appl, 2002, 276: 432-445.

[18] Miller S S, Mocanu P T. On some classes of first order differential subordinations[J]. Michigan Math J, 1985, 32: 185-195.

[19] Miller S S, Mocanu P T. Differential subordination: theory and applications, in: Series in Pure and Applied Mathematics, Vol 225[M]. New York: Marcel Dekker, 2000.

[20] Liu M S. On certain subclass of analytic functions[J]. J South China Normal Univ, 2002(4): 15-20.

[21] Miller S S, Mocanu P T. Subordinats of differential superordinations[J]. Complex Var Theory Appl, 2003, 48: 815-826.

[22] Shanmugam T N Ravichandran V, Sivasubramanian S. Differential sandwich theorems for subclasses of analytic functions[J]. Aust J Math Anal Appl, 2006(3): 1-11.

[23] Singh R, Singh S. Convolution properties of a class of starlike functions[J]. Proc Amer Math Soc, 1989, 106: 145-152.

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