DOI: 10.13656/j.cnki.gxkx.2010.01.010
广西科学 Guangxi Sciences 2010, 17(1): 32~ 35

On $\aleph_0$-*sn*-metric Spaces*
关于$\aleph_0$-*sn*-度量空间

WANG Pei[1], LI Zhong-min[2], LIU Shi-qin[3]
王 培[1],李忠民[2],刘士琴[3]

(1. Department of Mathematics and Computer Science, Yulin Normal University, Yulin, Guangxi, 537000, China; 2. The Second Middle School of Qingtian, Qingtian, Zhejiang, 323900, China; 3. Department of Mathematics, Hengshui University, Hengshui, Hebei, 253811, China)
(1.玉林师范学院数学与计算机科学系,广西玉林 537000; 2.青田二中,浙江青田 323900; 3.衡水学院数学系,河北衡水 253811)

Abstract A new concept of $\aleph_0$-*sn*-network is given, and demonstrate that it's weaker than $\aleph_0$-weak base through an example. Then some characterizations about $\aleph_0$-*sn*-network are given, and the relations among $\aleph_0$-weak base, $\aleph_0$-*sn*-network, *cs*-network, cs^*-network are discussed.

Key words $\aleph$-spaces, $\aleph_0$-spaces, $\aleph_0$-*sn*-network, cs^*-network.

摘要:先给出$\aleph_0$-*sn*-网的定义,并用实例说明此定义比$\aleph_0$-弱基更弱,并再讨论$\aleph_0$-*sn*-网的一些性质,得出$\aleph_0$-弱基、$\aleph_0$-*sn*-网、*cs*-网、cs^*-网之间的一些等价关系.

关键词: $\aleph$-空间 $\aleph_0$-空间 $\aleph_0$-*sn*-网 cs^*-网
中图法分类号: O189 文献标识码: A 文章编号: 1005-9164(2010)01-0032-04

Certain images of metric spaces have been studied extensively in the past years[1]. It plays a very important role in general topology. There are many excellent results which have been made by many people. C. Liu, S. Lin[2] defined $\aleph_0$-weak base. It is very well connect weak base and $\aleph_0$-weak base. C. Liu and S. Lin[2] prove that X is a quotient, countable-to-one image of a metric spaces if and only if X has a point-countable and point-countable $\aleph_0$-weak base. From this, they generalized the $\aleph_0$-weak base.

From their definitions, we easy know that a weak base is an $\aleph_0$-weak base. But $\aleph_0$-weak base may not be weak base. We will give a new concept as a generalization of $\aleph_0$-weak base, and some characterizations of it. Therefore we generalize C. Liu's and S. Lin's results.

收稿日期: 2009-06-08
修回日期: 2009-10-13
作者简介: 王 培(1980-),男,助教,主要从事一般拓扑学的研究。
* This project was supported by Guangxi Province Science Foundition (No. 0728035).

Throughout this paper, all spaces are regular T_1, all maps are continuous and onto, and N is the set of positive integer numbers. Sequence $\{x_n: n\in N\}$, sequence $\{P_n: n\in N\}$ of subsets and sequence $\{\mathscr{P}_n: n\in N\}$ of collections of subsets are abbreviated to $\{x_n\}$, $\{P_n\}$ and $\{\mathscr{P}_n\}$ respectively. For terms which are not defined here, please refer to reference [2] and related references.

1 Definitions

Definition 1.1 Let $\mathscr{B}$ be a family of subsets of a space X. $\mathscr{B}$ is said to be an $\aleph_0$-*sn*-network for X if $\mathscr{B}=\bigcup\{\mathscr{B}_x(n): x\in X, n\in N\}$ satisfies

(1) for each $x\in X, n\in N$, $\mathscr{B}_x(n)$ is a network at x. It is closed under finite intersections and $x\in\bigcap\mathscr{B}_x(n)$.

(2) L is a sequence converging to $x\notin L$ in X. Then there exist a subsequence L' of L and $n_0\in N$ such that L' is eventually in $B_x(n_0)$ for any $B_x(n_0)\in\mathscr{B}_x(n_0)$.

X is called $\aleph_0$-*sn*-weakly first-countable in the sense of Sirois-Dumais[3] if X has an $\aleph_0$-*sn*-network

$\mathscr{B}=\bigcup\{\mathscr{B}_x(n): x\in X, n\in N\}$, $\mathscr{B}_x(n)$ is countable for each $x\in X, n\in N$.

A space X is called $\aleph_0$-sn-metric space if X has a e-locally finite $\aleph_0$-sn-network. $\aleph_0$-sn-network is a generalization of $\aleph_0$-weak base and sn-network. It is very easy to see that $\aleph_0$-weak base doesn't imply sn-network. For example, S_k has a countable $\aleph_0$-weak base, but it does not have a countable sn-network (since Frechet space with a countable sn-network has a countable base). sn-network does not imply $\aleph_0$-weak base. For example, Uk (stone-cech compactification of k) is $\aleph_0$-sn-weakly first-countable since every convergent sequence is finite, but it is not $\aleph_0$-weakly first-countable, since it is not a sequential space (a sequential space in which every convergent sequence is finite is a discrete space).

Definition 1.2[4] Let X be a space, $P\subset X$ is called a sequential neighborhood of x in X, if each sequence converging to x in X is eventually in P.

Definition 1.3[5] Let $f: X\to Y$ be a sequentially quotient map if whenever $\{y_n\}$ is a convergent sequence in Y, there is a convergent sequence $\{x_k\}$ in X with each $x_k\in f^{-1}(y_{n_k})$.

Definition 1.4 Let $\mathscr{P}$ be a cover of X, then $\mathscr{P}$ is called a k-network for X if for any compact set K and for any open set U such that $K\subset\bigcup\mathscr{P}'\subset U$ for some finite $\mathscr{P}'\subset\mathscr{P}$. A space X is called $\aleph$-space if X has e-locally finite k-network.

2 Main results

In this section, we give some characterizations about $\aleph_0$-sn-network, and the relations among $\aleph_0$-weak base, $\aleph_0$-sn-network, cs-network, cs^*-network are discussed.

Lemma 2.1[2] X has a point-countable $\aleph_0$-weak base $\mathscr{B}=\bigcup\{\mathscr{B}_x(n): x\in X, n\in N\}$. L be a sequence converging to $x\in L$ in X. Then there exists a subsequence L' of L and $n_0\in N$ such that L' is eventually in $B_x(n_0, m)$ for any $m\in N$.

Remark 2.1 From the Lemma 2.1, it is easy to know that point-countable $\aleph_0$-weak base is $\aleph_0$-sn-network.

Remark 2.2 From the definition of $\aleph_0$-sn-network, it is easy to know that $\aleph_0$-sn-network is cs^*-network.

Theorem 2.1 Point-counatble $\aleph_0$-weak base is equivalent to a sequential space with a point-countable $\aleph_0$-sn-network.

Proof Sufficiency. Let $\mathscr{B}=\bigcup\{\mathscr{B}_x(n): x\in X, n\in N\}$ be a point-countable $\aleph_0$-weak base of a space X. From Lemma 2.1, it is easy to see that $\mathscr{B}$ is point-countable $\aleph_0$-sn-network of a space X. Since X has point-countable $\aleph_0$-weak base, then X is $\aleph_0$-weakly first countable space. Therefore X is a sequential space.

Necessity. X is a sequential space with a point-countable $\aleph_0$-sn-network $\mathscr{B}$. We shall prove that $\mathscr{B}$ is a Point-counatble $\aleph_0$-weak base. Since $\mathscr{B}$ is a point-countable $\aleph_0$-sn-network, for each $x\in X, n\in N$, $\mathscr{B}_x(n)$ is closed under finite intersections and $x\in\bigcap\mathscr{B}_x(n)$. If U is an open set of X and for each $x\in U$, $\mathscr{B}_x(n)$ is a network at x, there exists $n\in N$ such that $B_x(n)\subset U$. For any $x\in X, n\in N$, there exists $B_x(n)\in\mathscr{B}_x(n)$ such that $B_x(n)\subset U$. We only need to prove that U is an open set of X.

If U is not open in X, then U is not sequentially open set in X. There exists a convergent sequence L in $X\backslash U$ converging to a point $x\in U$. Since $\mathscr{B}$ is an $\aleph_0$-sn-network, there exist $n_0\in N$ and a subsequence L' of L such that L' is eventually in any elements of $\mathscr{B}_x(n_0)$. Therefore there exists $B_x(n_0)\in\mathscr{B}_x(n_0)$ such that $B_x(n_0)\subset U$. Then L' is eventually in U. U is a sequentially open set. So U is an open set of X.

Theorem 2.2 Let $\mathscr{B}=\bigcup\{\mathscr{B}_x(n): x\in X, n\in N\}$ be an $\aleph_0$-sn-network of a space X and A a subset of X. Then $\bigcup\{A\cap B_x(n): B_x(n)\in\mathscr{B}_x(n)\}$ is an $\aleph_0$-sn-network of A.

Proof For any $x\in A$ and $n\in N$. It is easy to see that $\{A\cap B_x(n): B_x(n)\in\mathscr{B}_x(n)\}$ is closed under finite intersections and $x\in(\bigcap\mathscr{B}_x(n))\cap A$.

For any $x\in A$ and $n\in N$, $\{A\cap B_x(n): B_x(n)\in\mathscr{B}_x(n)\}$ is a network at x in A. In fact, if U is open in A, A is a closed subset of X, there exists an open subset V of X such that $U=V\cap A$. For any $x\in U$ and $n\in N$, there is a $B_x(n)\in\mathscr{B}_x(n)$ such that $x\in B_x(n)\subset V$. Therefore $x\in B_x(n)\cap A\in\{B_x(n)\cap A: B_x(n)\in\mathscr{B}_x(n)\}$ and $B_x(n)\cap A\subset U$. We shall check that $\bigcup\{A\cap B_x(n): B_x(n)\in\mathscr{B}_x(n)\}$ is an $\aleph_0$-sn-network of A.

Let L be a sequence converging to $x\notin L$ in A.

Then L be a sequence converging to $x \notin L$ in X, since $\mathscr{B} = \bigcup \{\mathscr{B}_x(n): x \in X, n \in N\}$ be an $\aleph_0$-sn-network of a space X. There exists a subsequence L' of L and $n_0 \in N$ such that L' is eventually in $B_x(n_0)$ for any $B_x(n_0) \in \mathscr{B}_x(n_0)$, $L \subset A$, so $L' \subset A$, L' is eventually in $B_x(n_0) \cap A$ for any $B_x(n_0) \in \mathscr{B}_x(n_0)$. Therefore $\bigcup \{A \cap B_x(n): B_x(n) \in \mathscr{B}_x(n)\}$ is an $\aleph_0$-sn-network of A.

Lemma 2.2[1] Let $\mathscr{P}$ be a σ-hereditarily closure-preserving collection of subsets of a space X. If $\mathscr{P}$ is a cs^*-network, then $\mathscr{P}$ is a k-network of X.

Theorem 2.3 The following are equivalent for a space X.

(1) X is an $\aleph_0$-sn-metric space;

(2) X has a σ-discrete $\aleph_0$-sn-network;

(3) X has a σ-locally finite $\aleph_0$-sn-network;

(4) X is an $\aleph_0$-sn-weakly first-countable and $\aleph$-space.

Proof From the definition, it is easy know (1) $\leftrightarrow$ (3), and (2) $\rightarrow$ (3) is obvious.

We prove (3) $\rightarrow$ (4). Let $\mathscr{B}$ be a σ-locally finite $\aleph_0$-sn-network, then $\mathscr{B}$ is a point-countable $\aleph_0$-sn-network, X is an $\aleph_0$-sn-weakly first-countable space. From Remark 2.2, $\mathscr{B}$ is a cs^*-network, $\mathscr{B}$ be a σ-locally finite $\aleph_0$-sn-network, then $\mathscr{B}$ is a σ-hereditarily closure-preserving $\aleph_0$-sn-network. From Lemma 2.2, X has a σ-hereditarily closure-preserving k-network. So X be an $\aleph$-space.

Now we prove (4) $\rightarrow$ (2). X is an $\aleph$-space, by Theorem 4 in reference [6], we can assume that X has a σ-discrete cs-network $\mathscr{P}$, where $\mathscr{P}$ is closed under finite intersections. Let $\bigcup \{\mathscr{B}_x(n): x \in X, n \in N\}$ be an $\aleph_0$-sn-network of X. Then for each $x \in X, n \in N$, since X is an $\aleph_0$-sn-weakly first countable, $\mathscr{B}_x(n)$ is countable, here each $\mathscr{B}_x(n) = \{B_x(n,m): m \in N\}$ with $B_x(n, m+1) \subset B_x(n,m)$ for each $m \in N$. For each $n \in N$, let $\mathscr{P}_x(n) = \{P \in \mathscr{P}: B_x(n,m) \subset P$ for some $m \in N\}$. Then $\mathscr{P}_x(n)$ is closed under finite intersections. $\bigcup \{\mathscr{P}_x(n): x \in X, n \in N\}$ is a σ-discrete collection.

We shall prove that $\mathscr{P}_x(n)$ is a network of x in X. We only need to prove that there exist $m \in N, k \in N$ and fixing a neighborhood U of x in X such that $B_x(n,m) \subset P_k$, therefore $P_k \in \mathscr{P}_x(n)$ and $P_k \subset U$. If not, there is a neighborhood U of x in X, $P \not\subset U$ for each $P \in \mathscr{P}_x(n)$. Write $\{P \in \mathscr{P}: x \in P \subset U\} = \{P_k: k \in N\}$ such that $B(n,m) \not\subset P_k$ for any $m, k \in N$. Pick $x_{mk} \in B(n,m) \setminus P_k$ for each $m \geqslant k$. Let $y_i = x_{mk}$, where $i = k + m(m-1)/2$. Then the sequence $\{y_i\}$ converges to x in X because $\{B_x(n,m): m \in N\}$ is a decreasing network of x in X. Since $\mathscr{P}$ is a cs-network of X, there exists $k, j \in N$ such that $\{y_i: i \geqslant j\} \subset P_k$. Pick $i \geqslant j$ such that $y_i = x_{mk}$ for some $m \geqslant k$, then $x_{mk} \in P_k$, a contradiction.

Put $\mathscr{B} = \bigcup \{\mathscr{P}_x(n): x \in X, n \in N\}$. We shall prove that $\mathscr{B}$ is an $\aleph_0$-sn-network. Let L be a sequence converging to $x \notin L$ in X. Since $\bigcup \{\mathscr{B}_x(n): x \in X, n \in N\}$ be an $\aleph_0$-sn-network of a space X. There exists a subsequence L' of L and $n_0 \in N$ such that L' is eventually in $B_x(n_0, m)$ for any $B_x(n_0, m) \in \mathscr{B}_x(n_0)$. But $B_x(n_0,m) \subset P_x(n_0)$ for some $m \in N$. L' is eventually in $P_x(n_0)$ for any $P_x(n_0) \in \mathscr{P}_x(n_0)$. Therefore $\mathscr{B} = \bigcup \{\mathscr{P}_x(n): x \in X, n \in N\}$ is an $\aleph_0$-sn-network of X.

Y. Ge[7] give a equivalent characterization about sn-metric space. From his proof, we ask the following question naturally.

Question 2.1 Is a space X with σ-hereditarily closure-preserving $\aleph_0$-sn-network, $\aleph_0$-sn-weakly first-countable space and $\aleph$-space.

Theorem 2.4 X is a sequentially quotient, countable-to-one image of a metric space if and only if X has point-countable $\aleph_0$-sn-network.

Proof Necessity. Let $f: X \to Y$ be a sequentially quotient, countable-to-one map from a metric space M onto the space X. Let $\mathscr{B}$ be a point-countable base for M. For each $y \in M$, let $\mathscr{B}_y \subset \mathscr{B}$ be a countable, decreasing local base at y in M. Put $\mathscr{B}' = \{\mathscr{B}_y: y \in M\}$. Then $\mathscr{B}'$ is a point-countable family of M. Since f is a countable-to-one map, $f(\mathscr{B}')$ is a point-countable in X. We shall prove that $\bigcup f(\mathscr{B}')$ is an $\aleph_0$-sn-network.

For each $y \in M$, let $\mathscr{B}_y = \{B_{y,i}: i \in N\}$ with each $B_{y,i+1} \subset B_{y,i}$. For each $x \in X$, let $f^{-1}(x) = \{x_n: n \in N\}$. Because f is countable-to-one map, let $\mathscr{P}_x(n) = f(\mathscr{B}_{x_n})$. Then $\bigcup f(\mathscr{B}') = \bigcup \{\mathscr{P}_x(n): x \in X, n \in N\}$. Let U be open in X, for each $x \in U, n \in N$, $x_n \in f^{-1}(U)$, then $B_{x_n,i} \subset f^{-1}(U)$ for some $i \in N$, thus $f(B_{x_n,i}) \in \mathscr{P}_x(n)$ and $f(B_{x_n,i}) \subset U$. L be a sequence converging to x in X. f is a sequentially quotient map, there exists sequence S in M convergent to $x_{n_0} \in f^{-1}(x)$. For any $i \in N$, $B_{x_{n_0},i} \in \mathscr{B}_{x_{n_0}}$, from the Lemma 2.1, it is easy to see that $B_{x_{n_0},i}$ is a sequence neighborhood of x_{n_0}, S is eventually in $B_{x_{n_0},i}$ for any $i \in N$, so $f(S)$ is eventually in $f(B_{x_{n_0},i})$ for any $i \in N$ and $f(B_{x_{n_0},i}) \in \mathscr{P}_x(n_0)$, $f(S)$ is a subsequence of L and $f(S)$ is eventually in any elements of $\mathscr{P}_x(n_0)$. Therefore $f(\mathscr{B}')$ is an $\aleph_0$-sn-network.

Sufficiency. Let $\mathscr{B} = \bigcup \{\mathscr{B}_x(n): x \in X, n \in N\}$ be a point-countable $\aleph_0$-sn-network. Each $\mathscr{B}_x(n) = \{B_x(n,m): m \in N\}$ with each $B_x(n,m+1) \subsetneqq B_x(n,m)$ for each $m \in N$. Then any infinite subsequence $\mathscr{B}'_x$ of $\{B_x(n,m): m \in N\}$ is a network at x in X for each $x \in X$ and $n \in N$. We rewrite $\mathscr{B} = \{B_\alpha: \alpha \in I\}$. Endow I with discrete topology and let I_i be a copy of I for each $i \in N$. For convenience sake, two families $\{P_n: n \in N\}$ and $\{Q_m: m \in N\}$ of subsets of a space are said to be cofinal if there exists $n_0, m_0 \in N$ such that $P_{n_0+i} = Q_{m_0+i}$ for every $i \in N$. Put $M = \{\alpha = (\alpha_i) \in \prod_{i \in N} I_i: \{B_{\alpha_i}: i \in N\}$ is cofinal to $\mathscr{B}_{x(\alpha)}(n)$ for some $x(\alpha) \in X, n \in N$, $\{B_{\alpha_i}: i \in N\}$ is a network of $x(\alpha)\}$. Define $f: M \to X$ as $f(\alpha) = x(\alpha)$. It is easy to see that f is well-defined and onto because X is T_2 and each $\mathscr{B}_x(n)$ is a network of x in X for each $n \in N$. And $f(\alpha) = \bigcap_{i \in N} B_{\alpha_i}$ for each $\alpha = (\alpha_i) \in M$. Notice that $\mathscr{B}$ is point-countable, then f is countable-to-one. Also it is easy to prove f is continuous. We shall prove that f is sequentially quotient map.

Let $\mathscr{B} = \bigcup \{\mathscr{B}_x(n): x \in X, n \in N\}$ be a point-countable $\aleph_0$-sn-network. It is easy to see that L be a sequence converging to $x \notin L$ in X. Then there exists a subsequence L' of L and $n_0 \in N$ such that L' is eventually in $B_x(n_0,m)$ for any $m \in N$. For each $i \in N$ take $\alpha_i \in I_i$ with $B_{\alpha_i} = B_x(n_0,i)$. Let $\alpha = (\alpha_i)$, then $\alpha \in M$. For each $k \in N$, put $n_k = \min\{m \in N: x_k \notin B_x(n_0,m)\}$. Let $z_k = (\beta_i(k)) \in \prod_{i \in N} I_i$ as follows: if $i < n_k$, pick $\beta_i(k) \in I_i$ with $B_{\beta_i(k)} = B_x(n_0,i)$; otherwise pick $\beta_i(k) \in I_i$ such that $B_{\beta_i(k)} = B_{x_k}(1, i-n_k+1)$. Then $\{B_{\beta_i}(k): i \in N\}$ is cofinal to $\mathscr{B}_{x_k}(1)$, thus $z_k \in M$ and $f(z_k) = x_k$. On the other hand, for each $i \in N$ there exists $k_0 \in N$ such that $x_k \in B_x(n_0, i)$ for any $k \geqslant k_0$, because L' is eventually in $B_x(n_0,i)$. Then $i < n_k$ when $k \geqslant k_0$ by the definition of n_k, so $\beta_i(k) = \alpha_i$. It means that $\{B_{\beta_i}(k): i \in N\}$ converges to α_i in the discrete space I_i. Hence z_k converges to α in M. Therefore, f is sequentially quotient map.

Remark 2.3 $\aleph_0$-sn-network may not be sn-network. If not, point-countable $\aleph_0$-sn-network is point-countable sn-network. A space with point-countable sn-network is sequential space, then point-countable sn-network is point-countable weak base. So point-countable $\aleph_0$-sn-network is point-countable weak base[8]. This is not true.

Remark 2.4 $\aleph_0$-sn-network may not be cs-network. Every quotient finite-to-one image of a locally compact metric space does not have a point-countable cs-network[9]. But C. Liu proof that it has point-countable $\aleph_0$-weak base[2]. Then X has point-countable $\aleph_0$-sn-network.

Acknowledgement

The author would like to thank the professor C. Liu for his suggestions.

References

[1] Lin S. Generalized metric spaces and mapping [M]. Beijing: Chinese Science Press, 1995.

[2] Liu C, Lin S. On countable-to-one maps [J]. Topology Appl, 2007, 154(2): 449-454.

[3] Sirois-Dumais R. Quasi-and weakly quasi-first-countable spaces [J]. Topology Appl, 1980, 11(3): 223-230.

[4] Lin S. Sequence-covering maps of metric spaces [J]. Topology Appl, 2000, 109(3): 301-314.

[5] Boone J R, Siwiec F. Sequentially quotient mappings [J]. Czech Math J, 1976, 26: 174-182.

[6] Foged L. Characterizations of $\aleph$-space [J]. Pacific J Math, 1984, 110: 59-63.

[7] Ge Y. On space with a σ-locally finite universal cs-network [J]. Questions Answers in General Topology, 2000, 18(1): 93-96.

[8] Arhangel′skill A V. Mapping and spaces [J]. Russian Math Surveys, 1966(21): 115-162.

[9] Lin S, Yoshio Tanaka. Point-countable k-networks, closed maps, and related results [J]. Topology Appl, 1994, 59: 79-86.

(责任编辑: 尹　闯)