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The Genus of the Zero-divisor Graph of $Z_n[i]$*
模 n 高斯整环 $Z_n[i]$ 的零因子图的类数

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Abstract The positive integers n such that the genus of the zero-divisor graph of $Z_n[i]$ is 0, 1, 2, 3, 4, or 5 are completely determined.

Key words genus of a graph, zero-divisor graph, the ring of Gaussian integers modulo n.

摘要:完全决定了模 n 高斯整环 $Z_n[i]$ 的零因子图的类数分别为 0, 1, 2, 3, 4, 5的情况.

关键词:图的类数 零因子图 模 n 高斯整数环
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Throughout this paper it is assumed that all rings are commutative with identity. Let R be a commutative ring with identity, $Z[R]$ denotes its set of zero-divisors. For a ring R, we associate a simple graph $\Gamma(R)$ to R with vertices $Z(R)^* = Z(R) - \{0\}$, and for distinct $x, y \in Z(R)^*$, the vertices x and y are adjacent if and only if $xy = 0$. An element a in R is called a unit if there exists an element b of R such that $ab = 1$. We say that two elements a and b are associates if there exists a unit u such that $a = ub$. For an element t of R, the associate class of t, denoted by A_t, is the set of all associate elements of t.

A simple graph G is an ordered pair of disjoint sets (V, E) such that $V = V(G)$ is the vertex set of G and $E = E(G)$ is the edge set of G. Given $v \in V$, the degree of v, denoted by deg (v), is the number of edges of G incident with v. Let $V' \subseteq V(G)$, then $G - V'$ is the subgraph of G obtained by deleting the vertices in V' and all edges incident with them. If $V' = \{v \in V \mid \deg(v) = 1\}$, then the subgraph $G - V'$, denoted by $\tilde{G}$, is called the reduction of G. A bipartite graph G is a graph such that its vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 and that each edge $e \in E(G)$ joins a vertex of V_1 to a vertex of V_2. In particular, if $E(G)$ consists of all edges joining V_1 with V_2, then it is called a complete bipartite graph and is denoted by $K_{m,n}$, where $|V_1| = m$ and $|V_2| = n$. A graph G in which each pair of distinct vertices is joined by an edge is called a complete graph, denoted by K_n, where $n = |V(G)|$.

A surface is said to be of genus g if it is topologically homemorphic to a sphere with g handles. A graph G can be drawn without crossing on a compact surface of genus g, but not on one of genus $g - 1$, is called a graph of genus g. We write $\gamma(G)$ for the genus of the graph G. Recently, this subject has been studied extensively in reference [1~6]. In this paper, we completely determine the positive integers n such that the genus of the zero-divisor graph of $Z_n[i]$ is 0, 1, 2, 3, 4, or 5, where $Z_n[i] = \{a + bi \mid a, b \in Z_n\}$ is the ring of Gaussian integers modulo n.

1 Some lemmas

Lemmas 1.1[7] $\gamma(K_n) = \{\frac{1}{12}(n-3)(n-4)\}$ for $n \geqslant 3$, where the notation $\{x\}$ represents the minimum interger that is greater than or equal to x.

Lemmas 1.2[7] $\gamma(K_{m,n}) = \{\frac{1}{4}(m-2)(n-2)\}$ for $m, n \geqslant 2$, where the notation $\{x\}$ represents the minimum interger that is greater than or equal to

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x.

Lemmas 1.3[6] $\gamma(H)\leqslant\gamma(G)$ for any subgraph H of G, and $\gamma(\widetilde{G})=\gamma(G)$, where $\widetilde{G}$ is the reduction of G.

Lemmas 1.4[8] If G is a connected graph of genus γ such that every face is a triangle, then $q=3(p-2+2\gamma)$, where $p=|V(G)|$ and $q=|E(G)|$.

Lemmas 1.5 Let $R=Z_{p^k}[i]$ and p a prime. Then the associate classes $A_{p^t}(1\leqslant t<k)$ of R state the following (1) If $p=2$, then $|A_{2^t}|=2^{2(k-t)-1}$; (2) If $p\equiv 3(\mathrm{mod}4)$, then $|A_{p^t}|=p^{2(k-t-1)}(p^2-1)$; (3) If $p\equiv 1(\mathrm{mod}4)$, then $|A_{p^t}|=p^{2(k-t-1)}(p-1)^2$.

proof Note that, $R=Z_{p^k}[i]=\{a+bi|a,b\in Z_{p^k}\}$. Let $S_{p^t}=\{sp^t|s\in Z_{p^k},gcd(s,p)=1$ and $1\leqslant s<p^{k-t}\}$ where $t\in\{1,2,\cdots,k-1\}$. Then one can divide the elements of Z_{p^k} into $k-1$ sets. The use of the Euler's phi-function gives the size of the set S_{p^t}'s and one will get $|S_{p^t}|=p^{k-t}-p^{k-(t+1)}$.

For any $a+bi\in U(R)$, let $p^t=(a+bi)p^t$, then we have

$$\begin{cases} p^t(a-1)\equiv 0(\mathrm{mod}\ p^k), & (1)\\ p^t b\equiv 0(\mathrm{mod}p^k). & (2)\end{cases}\qquad(1.1)$$

From congruence equation (1.1), we get $a-1\in\{0\}\cup S_{p^{k-1}}\cup S_{p^{k-2}}\cup\cdots\cup S_{p^{k-t}}$, $b\in\{0\}\cup S_{p^{k-1}}\cup S_{p^{k-2}}\cup\cdots\cup S_{p^{k-t}}$, so the numbers of solutions of these congruence equation (1) and (2) in formula (1.1) are both: $|\{0\}\cup S_{p^{k-1}}\cup S_{p^{k-2}}\cup\cdots\cup S_{p^{k-t}}|=|\{0\}|+|S_{p^{k-1}}|+|S_{p^{k-2}}|+\cdots+|S_{p^{k-t}}|=1+(p-1)+(p^2-p)+\cdots+(p^t-p^{t-1})=p^t$, then the number of solutions of formula (1.1) is p^{2t}. Hence, we have $|A_{p^t}|=|U(Z_{p^k}[i])|/p^{2t}$.

(1) If $p=2, R=Z_{2^k}[i]$, from Theorem 3.1(1) in reference [9], we have $|U(Z_{2^k}[i])|=2^{2k-1}$. Therefore $|A_{p^t}|=2^{2k-1}/2^{2t}=2^{2(k-t)-1}$.

(2) If $p\equiv 3(\mathrm{mod}4), R=Z_{p^k}[i]$, then from Theorem 3.1(2) in reference [9], we have $|U(Z_{p^k}[i])|=p^{2k}-p^{2k-2}$. Therefore $|A_{p^t}|=(p^{2k}-p^{2k-2})/p^{2t}=p^{2(k-t-1)}(p^2-1)$.

(3) If $p\equiv 1(\mathrm{mod}4), R=Z_{p^k}[i]$, then from Theorem 3.1(3) in reference [9], we have $|U(Z_{p^k}[i])|=p^{2k}-2p^{2k-1}+p^{2k-2}$. Therefore $|A_{p^t}|=(p^{2k}-2p^{2k-1}+p^{2k-2})/p^{2t}=p^{2(k-t-1)}(p-1)^2$.

2 The genus of $\Gamma(Z_{p^k}[i])$

Theorem 2.1 Let $R=Z_{2^k}[i]$. Then $\gamma(\Gamma(R))\leqslant 5$ if and only if $R=Z_2[i],Z_4[i]$, or $Z_8[i]$. Further more, $\gamma(Z_2[i])=\gamma(Z_4[i])=0$, and $\gamma(Z_8[i])=3$.

Proof If $k=1,2$, by [10. Theorem 4.5], we have $\gamma(Z_2[i])=\gamma(Z_4[i])=0$;

If $k=3$, then $n=8=(-i)^3(1+i)^6$, we can divide the nonzero zero-divisors of $Z_8[i]$ into the following 5 sets $A_{(1+i)},A_{(1+i)^2},A_{(1+i)^3},A_{(1+i)^4},A_{(1+i)^5}$.

For any $(a+bi)\in Z_8[i]$, let $(1+i)^t=(a+bi)(1+i)^t$, $t\in\{1,2,\cdots,5\}$, then we have

(1) If t is odd, then

$$\begin{cases} 2^{\frac{t-1}{2}}(a-b-1)\equiv 0(\mathrm{mod}\ 8),\\ 2^{\frac{t-1}{2}}(a+b-1)\equiv 0(\mathrm{mod}\ 8).\end{cases}\qquad(2.1)$$

(2) If t is even, then

$$\begin{cases} 2^{\frac{t}{2}}(a-1)\equiv 0(\mathrm{mod}\ 8),\\ 2^{\frac{t}{2}}b\equiv 0(\mathrm{mod}\ 8).\end{cases}\qquad(2.2)$$

By solving congruence equation (2.1) and (2.2), we get the orders of the associate classes of $Z_8[i]$: $|A_{(1+i)}|=2^4$, $|A_{(1+i)^2}|=2^3$, $|A_{(1+i)^3}|=2^2$, $|A_{(1+i)^4}|=2$, and $|A_{(1+i)^5}|=1$. Since $\forall x\in A_{(1+i)}$, $\deg(x)=1$, and every other vertex in $\Gamma(Z_8[i])$ has degree $\geqslant 2$, by Lemma 1.3, we have $\widetilde{\Gamma(Z_8[i])}=\Gamma(Z_8[i])-A_{(1+i)}$, and $\gamma(\widetilde{\Gamma(Z_8[i])})=\gamma(\Gamma(Z_8[i]))$. Since $\widetilde{\Gamma(Z_8[i])}$ is a connected graph, and every face is a triangle, $|V(\widetilde{\Gamma(Z_8[i])})|=15$, $|E(\widetilde{\Gamma(Z_8[i])})|=45$, by Lemma 1.4, we have $\gamma(\widetilde{\Gamma(Z_8[i])})=3$. Therefore $\gamma(\Gamma(Z_8[i]))=3$.

If $k=4$, then $n=16=(-i)^4(1+i)^8$. Similarly, we have $|A_{(1+i)^3}|=2^4$, $|A_{(1+i)^5}|=2^2$, then $K_{4,16}\subseteq\Gamma(Z_{2^4}[i])$. Therefore $\gamma(\Gamma(Z_{2^4}[i]))\geqslant\gamma(K_{4,16})=7>5$ by Lemma 1.2.

If $k\geqslant 5$, by Lemma 1.4, we have $|A_{2^{k-2}}|=2^3$, $|A_{2^{k-3}}|=2^5$, then $K_{8,32}\subseteq\Gamma(Z_{2^k}[i])$. Therefore $\gamma(\Gamma(Z_{2^k}[i]))\geqslant\gamma(K_{8,32})=45>5$ by Lemma 1.2.

Theorem 2.2 Let $R=Z_{p^k}[i]$, where $p\equiv 3(\mathrm{mod}\ 4)$ is a prime. Then $\gamma(\Gamma(R))\leqslant 5$ if and only if $k=1$ or $k=2$ and $p=3$; Further more, for $k=1$, $\Gamma(Z_p[i])$ is an empty graph, and $\gamma(\Gamma(Z_9[i]))=2$.

proof If $k=1$, then $Z_p[i]$ is a finite field and in this case, $\Gamma(Z_p[i])$ is an empty graph.

If $k=2$, $\Gamma(Z_{p^2}[i])$ is complete graph by [11. Theorem 15] and $\Gamma(Z_{p^2}[i])=K_{p^2-1}$, then for $p=3$, $\gamma(\Gamma(Z_{3^2}[i]))=\gamma(K_8)=2$, and for $p>3$, $\gamma(\Gamma(Z_{p^2}[i]))>5$ by Lemma 1.1.

If $k\geqslant 3$, then there are at least $q^2(q^2-1)$ elements in $A_{q^{k-2}}$ and at least q^2-1 elements in $A_{q^{k-1}}$, so that $K_{8,72}\subseteq\Gamma(Z_{q^k}[i])$, therefore $\gamma(\Gamma(Z_{q^k}[i]))\geqslant\gamma(K_{8,72})=105>5$ by Lemma 1.2.

Theorem 2.3 Let $R=Z_{p^k}[i], p\equiv 1(\mathrm{mod}\ 4)$ a prime. Then $\gamma(\Gamma(R))\leqslant 5$ if and only if $R=Z_5[i]$; Further more, $\gamma(\Gamma(Z_5[i]))=1$.

Proof If $k = 1$, $Z_p[i]$ is a complete bipartite graph by [11. Theorem 17] and $\Gamma(Z_p[i]) = K_{p-1,p-1}$, then for $p = 5$, $\gamma(\Gamma(Z_5[i])) = \gamma(K_{4,4}) = 1$, and for $p > 5$, $\gamma(\Gamma(Z_p[i])) > 5$ by Lemma 1.2.

If $k = 2$, then there are $(p-1)^2$ elements in A_p, so that $K_{16} \subseteq \Gamma(Z_{p^2}[i])$. Therefore $\gamma(\Gamma(Z_{p^2}[i])) \geqslant \gamma(K_{16}) = 13 > 5$ by Lemma 1.1.

If $k \geqslant 3$, then there are $p^2(p-1)^2$ elements in $A_{p^{k-2}}$ and there are $(p-1)^2$ elements in $A_{p^{k-1}}$, so that $K_{16,400} \subseteq \Gamma(Z_{p^k}[i])$. Therefore $\gamma(Z_{q^k}[i]) \geqslant \gamma(K_{16,400}) = 1393 > 5$ by Lemma 1.2.

3 The genus of $\Gamma(Z_n[i])$

Theorem 3.1 Let $R = Z_n[i]$, $n = 2^m p_1^{k_1} \cdots p_s^{k_s} q_1^{l_1} \cdots q_t^{l_t}$, where $p_i \equiv 1 (\mathrm{mod}\ 4)$ and $q_i \equiv 3(\mathrm{mod}\ 4)$ are prime numbers, $k_i \geqslant 1$, $l_j \geqslant 1$, $i = 1, \cdots, s$; $j = 1, \cdots, t$; $m \geqslant 0$. Then $\gamma(Z_n[i]) \leqslant 5$ if and only if $R = Z_2[i], Z_4[i], Z_5[i], Z_6[i], Z_8[i], Z_9[i]$ or $R = Z_p[i]$ and $p \equiv 3(\mathrm{mod}\ 4)$ is a prime; Further more, $\gamma(Z_2[i]) = \gamma(Z_4[i]) = 0$, $\gamma(Z_5[i]) = 1$, $\gamma(Z_6[i]) = \gamma(Z_9[i]) = 2$ and $\gamma(Z_8[i]) = 3$.

Proof By [9. Theorem 3.2], we have $Z_n[i] \cong Z_{2^m}[i] \oplus Z_{p_1^{k_1}}[i] \oplus \cdots \oplus Z_{p_s^{k_s}}[i] \oplus Z_{q_1^{l_1}}[i] \oplus \cdots \oplus Z_{q_t^{l_t}}[i]$.

Case 1. Assume $s \geqslant 2$. In this case, let $u_i = (0, a, 0, \cdots, 0)$, $v_j = (0, 0, b, \cdots, 0) \in \Gamma(R)$, $a \in Z_{p_1^{k_1}}[i]$, $b \in Z_{p_2^{k_2}}[i]$, and $|Z_{p_2^{k_2}}[i]| > |Z_{p_1^{k_1}}[i]| \geqslant 25$. Since $u_i v_j = 0$ for every i, j, $K_{24,24} \subseteq \Gamma(R)$, then $\gamma(\Gamma(R)) \geqslant \gamma(K_{24,24}) = 121 > 5$ by Lemma 1.2.

Case 2. Assume $s = 1$.

(a) If $t \geqslant 1$, let $u_i = (0, a, 0, \cdots, 0)$, $v_j = (0, 0, b, \cdots, 0) \in \Gamma(R)$, $a \in Z_{p_1^{k_1}}[i]$, $b \in Z_{q_1^{l_1}}[i]$ and $|Z_{p_1^{k_1}}[i]| \geqslant 25$, $|Z_{q_1^{l_1}}[i]| \geqslant 9$. Since $u_i v_j = 0$ for every i, j, $K_{8,24} \subseteq \Gamma(R)$, then $\gamma(\Gamma(R)) \geqslant \gamma(K_{8,24}) = 33 > 5$ by Lemma 1.2.

(b) If $t = 0$, $m \geqslant 1$, let $u_i = (a, 0)$, $v_j = (0, b)$, $a \in Z_{2^m}[i]$, $b \in Z_{p_1^{k_1}}[i]$ and $|Z_{2^m}[i]| \geqslant 4$, $|Z_{p_1^{k_1}}| \geqslant 25$. Since $u_i v_j = 0$ for every i, j, $K_{3,24} \subseteq \Gamma(R)$, then $\gamma(\Gamma(R)) \geqslant \gamma(K_{3,24}) = 6 > 5$ by Lemma 1.2.

(c) If $t = 0$, $m = 0$, then $n = p_1^{k_1}$, $\gamma(Z_5[i]) = 1$ by Theorem 2.3.

Case 3. Assume $s = 0$.

(a) If $t \geqslant 2$, let $u_i = (0, a, 0, \cdots, 0)$, $v_j = (0, 0, b, \cdots, 0) \in \Gamma(R)$, $a \in Z_{q_1^{l_1}}[i]$, $b \in Z_{q_2^{l_2}}[i]$ and $|Z_{q_2^{l_2}}[i]| > |Z_{q_1^{l_1}}[i]| \geqslant 3$. Since $u_i v_j = 0$ for every i, j, $K_{8,8} \subseteq \Gamma(R)$, then $\gamma(\Gamma(R)) \geqslant \gamma(K_{8,8}) = 9 > 5$ by Lemma 1.2.

(b) If $t = 1$, $m \geqslant 2$, let $u_i = (a, 0)$, $v_j = (0, b)$, $a \in Z_{2^m}[i]$, $b \in Z_{q_1^{l_1}}[i]$ and $|Z_{2^m}[i]| \geqslant 16$, $|Z_{q_1^{l_1}}[i]| \geqslant 9$. Since $u_i v_j = 0$ for every i, j, $K_{8,15} \subseteq \Gamma(R)$, then $\gamma(\Gamma(R)) \geqslant \gamma(K_{8,15}) = 20 > 5$ by Lemma 1.2.

(c) If $t = 1$, $m = 1$, $l_1 \geqslant 2$, let $u_i = (a, 0)$, $v_j = (0, b)$, $a \in Z_{2^m}[i]$, $b \in Z_{q_1^{l_1}}[i]$, $|Z_{2^m}[i]| = 4$, $|Z_{q_1^{l_1}}[i]| \geqslant 81$. Since $u_i v_j = 0$ for every i, j, $K_{3,80} \subseteq \Gamma(R)$, then $\gamma(\Gamma(R)) \geqslant \gamma(K_{3,80}) = 20 > 5$ by Lemma 1.2.

(d) If $t = 1$, $m = 1$, $l_1 = 1$, then $n = 2q_1$. If $q_1 > 3$, let $u_i = (a, 0)$, $v_j = (0, b)$, $a \in Z_{2^m}[i]$, $b \in Z_{q_1}[i]$, and $|Z_{2^m}[i]| = 4$, and $|Z_{q_1}[i]| \geqslant 49$. Since $u_i v_j = 0$ for every i, j, $K_{3,48} \subseteq \Gamma(R)$, then $\gamma(\Gamma(R)) \geqslant \gamma(K_{3,48}) = 12 > 5$ by Lemma 1.2. If $q_1 = 3$, $Z_6[i] = Z_2[i] \oplus Z_3[i]$, then $\gamma(\Gamma(Z_6[i])) = \gamma(K_{3,8}) = 2$ by Lemma 1.2.

(e) If $t = 1$, $m = 0$, then $n = q_1^{l_1}$. $\gamma(Z_9(i)) = 2$ by theorem 2.2.

(f) If $t = 0$, $m \neq 0$, then $n = 2^m$. $\gamma(Z_8(i)) = 3$ by theorem 2.1.

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