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C-supplemented and SS-quasinormal Subgroups of Finite Groups*

有限群的 C-补和 SS-拟正规子群

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Abstract Let G be a finite group, p the smallest prime dividing the order of G and P a Sylow p-subgroup of G. We will show that if every members of $M(P)$ is either SS-quasinormal or C-supplemented in G, then G is p-nilpotent.

Key words SS-quasinormal subgroup, C-supplemented subgroup, Maximal subgroup, p-nilpotent, solvable group

摘要:用 Sylow 子群的极大子群 SS-拟正规和 C-补性质来刻画一些群系.证明:若有限群 G的所有极大 Sylow 子群是 SS-拟正规的或者 C-补的,那么 G是 p-幂零的,其中 G是有限群,p是 G的最小素数阶划分.

关键词: SS-拟正规子群 C-补子群 极大子群 p-幂零 可解群
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All groups considered in this paper will be finite. If H is a subgroup of the group G, then H_G denotes the normal core of H in G, the largest normal subgroup of G which is contained in H. Also, $\Phi(G)$ is the Frattini subgroup of G. $M(G)$ denotes the set of all maximal subgroups of all Sylow subgroups of G.

A class F of finite groups is called a formation if it has the following properties:

(1) if $G \in F$ and $N \trianglelefteq G$, then $G/N \in F$.

(2) if $G/N_i \in F (i = 1, 2)$, then $G/N_1 \cap N_2 \in F$.

If in addition that, $G/\Phi(G) \in F$ implies $G \in F$, then F is called saturated. It is well known that the class of all supersovable groups is a saturated formation denoted as U.

A subgroup H of G is called S-quasinormal in G if H permutes with every Sylow subgroup of G (i. e, $HS = SH$ for any Sylow subgroup S of G). This concept was introduced by O. H. Kegel in 1962 and was investigated by many authors. Recently, Ballester-Bolinches and Pedraza-Aguilera generalized this concept to S-quasinormally embedded subgroups. A subgroup H of G is called S-quasinormally embedded in G, provided every Sylow subgroup of H is a Sylow subgroup of some S-quasinormal subgroup K of G. H is called C-supplemented in G if there exists a subgroup N of G such that $G = HN$ and $H \cap N \leqslant H_G = Core_G(H)$.

In 2008, Li S and Shen Z[1] introduced the concept of SS-quasinormality. A subgroup H of a group G is said to be an SS-quasinormal subgroup of G, if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B. In this paper, we will show that if every member of $M(P)$ is either SS-quasinormal or C-supplemented in G, then G is p-nilpotent. As applications, some further results are obtained.

1 Preliminaries

In this section, we collect some known results which are needed in this article.

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Theorem 1.1[2] Let p be a prime dividing the order of G and P is a Sylow subgroup of G. If every maximal subgroup of P is either S-quasinormally embedded or C-supplemented in G and $(|G|, p-1) = 1$, then G is p-nilpotent.

Theorem 1.2[3] Let F be a saturated formation containing U, G is a group with a normal subgroup H such that $G/H \in F$. Then $G \in F$ if one of the following holds:

(1) all maximal subgroups of all Sylow subgroups of H are either S-quasinormal or C-supplemented in G;

(2) all maximal subgroups of all Sylow subgroups of $F^*(H)$ are either S-quasinormal or C-supplemented in G.

Lemma 1.1[1] Suppose H is SS-quasinormal in a group G, $K \leqslant G$ and N a normal subgroup of G.

(1) If $H \leqslant K$, then H is SS-quasinormal in K.

(2) HN/N is SS-quasinormal in G/N.

(3) If $N \leqslant K$ and K/N is SS-quasinormal in G/N, then K is SS-quasinormal in G.

(4) If K is quasinormal in G, then HK is SS-quasinormal in G.

Lemma 1.2[2] Let G be a group.

(1) If H is C-supplemented in G, $H \leqslant K \leqslant G$, then H is C-supplemented in K.

(2) Let $K \trianglelefteq G$ and $H \leqslant K \leqslant G$, then H is C-supplemented in G if and only if H/K is C-supplemented in G/K.

(3) Let π be a set of primes, H a π-subgroup of G and N a normal π'-subgroup of G. If H is C-supplemented in G, then HN/N is C-supplemented in G/N.

Lemma 1.3[1] Let H be a nilpotent subgroup of G. Then the following statements are equivalent.

(1) H is S-quasinormal in G.

(2) $H \leqslant F(G)$ and H is SS-quasinormal in G.

(3) $H \leqslant F(G)$ and H is S-quasinormally embedded in G.

Lemma 1.4[3] Let G be a group and p a prime dividing $|G|$ with $(|G|, p-1) = 1$.

(1) If N is normal in G of order p, then $N \leqslant Z(G)$.

(2) If G has cyclic Sylow p-subgroup, then G is p-nilpotent.

(3) If $M \leqslant G$ and $|G:M| = p$, then $M \trianglelefteq G$.

Lemma 1.5[4] Let G be a group and M is a subgroup of G.

(1) If $M \trianglelefteq G$, then $F^*(M) \leqslant F^*(G)$.

(2) $F^*(G) \neq 1$, if $G \neq 1$; in fact $F^*(G)/F(G) = \mathrm{soc}(F(G)C_G(F(G))/F(G))$.

(3) $F^*(F^*(G)) = F^*(G) \geqslant F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.

2 Main results

Theorem 2.1 Let p be a prime dividing the order of G and P a Sylow p-subgroup of G. If every maximal subgroup of P is either SS-quasinormal or C-supplemented in G and $(|G|, p-1) = 1$, then G is p-nilpotent.

Proof If all maximal subgroups of P are C-suplemented in G, then G is p-nilpotent by Theorem 1.1. Hence there exists a maximal subgroup P_1 of P such that P_1 is SS-quasinormal in G. Firstly, fix an $H \in M(P)$ such that H is SS-quasinormal in G.

Now we prove that there exists a Hall p'-subgroup K of G such that HK is a subgroup of index p in G.

By conditions, there is a subgroup $B \leqslant G$ such that $G = HB$ and $HX = XH$ for all $X \in \mathrm{Syl}(B)$, and $H \cap B$ is of index p in B_p, a Sylow p-subgroup of B containing $H \cap B$. Thus $S \nsubseteq H$ and $S \cap H = B \cap H$ for all $S \in \mathrm{Syl}_p(B)$. So $B \cap H = \bigcap_{b \in B}(S^b \cap H) \leqslant \bigcap_{b \in B} S^b = O_p(B)$.

We claim that B has a Hall p'-subgroup. Because $|O_p(B) : B \cap H| = p$ or 1, it follows that $|B/O_p(B)|_p = p$ or 1. As $(|G|, p-1) = 1$, then $B/O_p(B)$ is p-nilpotent by Lemma 1.4(2), and hence B is p-solvable. So B has a Hall p'-subgroup. Thus the claim holds. Now, let K be a Hall p'-subgroup of B. $\pi(K) = \{p_2, \cdots, p_s\}$ and $P_i \in \mathrm{Syl}_{p_i}(K)$. By the conditions, H is SS-quasinormal in G, so H permute with subgroup $\langle P_2, \cdots, P_s \rangle = K$ and $HK \leqslant G$. Moreover, $|G : HK| = p$ as desired.

Now, for every $H_i \in M(P)$ (H_i is SS-quasinormal in G), there exists a Hall p'-subgroup K_i of G such that $M_i = H_i K_i$, which is a subgroup of index p in G. As $(|G|, p-1) = 1$, by Lemma 1.4, $M_i \trianglelefteq G$. Obviously, H_i is S-quasinormally embedded in G. Thus every maximal subgroup of G is either C-supplemented or S-quasinormally embedded in G. By Theorem 1.1, G is p-nilpotent.

Corollary 2.1 Let p be the smallest prime dividing the order of G and P a Sylow p-subgroup of G. If every maximal subgroup of P is either SS-quasinormal or C-supplemented in G, then G is p-nilpotent.

Corollary 2.2 Suppose that G is a group. If every member of $M(G)$ is either SS-quasinormal or C-supplemented in G, then G has Sylow tower of supersolvable type.

Proof It is clear that $(|G|, p-1) = 1$, if p is the smallest prime dividing $|G|$. By the hypothesis, every member of $M(G)$ is either SS-quasinormal or C-supplemented in G, so G satisfies the condition of Theorem 2.1, and hence G is p-nilpotent. Let U be the normal p-complement of G, then U satisfies the condition by induction, hence G possesses Sylow tower property of supersolvable type.

Theorem 2.2 Let F be a saturated formation containing U. Suppose that G is a group with a normal subgroup H such that $G/H \in F$. If for every prime p dividing $|H|$ and $P \in Syl_p(H)$, every member of $M(P)$ is either SS-quasinormal or C-supplemented in G, then $G \in F$.

Proof Assume that the theorem is not true and let G be a minimal counter-example.

(1) H has minimal normal subgroup H_1, $H_1 \leqslant Q \leqslant H$, $Q \in Syl_q(H)$ and q is the largest prime in $c(H)$.

Obviously, H satisfies the condition of Corollary 2.2, so H possesses Sylow tower property of supersolvable type. Let q is the largest prime dividing $|H|$ and Q is a Sylow q-subgroup of H, then $Q4\ H$, so H has minimal normal subgroup H_1, $H_1 \leqslant Q$ and H_1 is an elementary abelian q-group, as desired.

(2) $G/H_1 \in F$, $H_1 \leqslant\!\!/\, H(G)$, $H_1 = Q \in Syl_q(H)$.

Obviously, $G/H_1 \in F$. Since F is a saturated formation, so H_1 is the unique minimal normal subgroup of G containing in H, $H_1 \leqslant\!\!/\, H(G)$. Moreover, $H_1 = F(H)$. Since H is solvable, so $C_H(H_1) \leqslant F(H)$ and $C_H(H_1) = H_1 = F(H)$. Since $Q4\ H$, $Q \leqslant F(H)$, thus $H_1 = Q \in Syl_q(H)$.

(3) The final contradiction.

For any maximal subgroup Q_1 of Q, Q_1 is either SS-quasinormal or C-supplemented in G by (2) and the hypothesis. Thus Q_1 is either S-quasinormal or C-supplemented in G by Lemma 1.3. Hence $G \in F$ by Theorem 1.2. We get the final contradiction.

Corollary 2.3 Let G be a group, H a normal subgroup of G such that G/H is supersolvable. If all maximal subgroups of all Sylow subgroups of H are either SS-quasinormal or C-supplemented in G, then G is supersolvable.

Theorem 2.3 Let F be a saturated formation containing U. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in F$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are either SS-quasinormal or C-supplemented in G, then $G \in F$.

Proof As H is solvable, by Lemma 1.5, $F(H) = F^*(H)$. Let $P_1 \in M(F(H))$, if P_1 is SS-quasinormal in G, then P_1 is S-quasinormal in G by Lemma 1.3. Applying Theorem 1.2(2), we can get $G \in F$.

Theorem 2.4 Let F be a saturated formation containing U. Suppose that G is a group with a normal subgroup H such that $G/H \in F$. If every member of $M(F^*(H))$ is either SS-quasinormal or C-supplemented in G, then $G \in F$.

Proof Suppose that the theorem is false and let G be a minimal counter-example.

Case 1 $F = U$.

Let G be a minimal counter-example.

(1) Every proper normal subgroup N of G containing $F^*(H)$ is supersolvable.

Since $N/N \cap H \cong NH/H$ is supersolvble, we get $F^*(H) = F^*(F^*(H)) \leqslant F^*(N \cap H) \leqslant F^*(H)$ by Lemma 1.5. So $F^*(H) = F^*(N \cap H)$ and N, $N \cap H$ satisfy the hypothesis of the theorem. Hence N is supersolvable by the minimal choice of G.

(2) $H = G$ and $1 \neq F^*(G) = F(G) < G$.

If $H < G$, then H is supersolvable as H contains $F^*(H)$ and $F^*(H) = F(H)$, it follows that G is supersolvable by Theorem 2.3, a contradiction.

If $F^*(G) = G$, then G is supersolvable by applying Corollary 2.3, a contradiction. Thus $F^*(G) < G$, it is supersolvable by (1), so $F^*(G) = F(G) \neq 1$ by Lemma 1.5.

(3) The final contradiction.

For any Sylow subgroup P of $F^*(H)$ and for any maximal subgroup P_1 of P, P_1 is either SS-quasinormal or C-supplemented in G by the hypothesis. As $P_1 \leqslant F(G)$, so P_1 is either S-quasinormal or C-supplemented in G by Lemma 1.3. Applying Theorem 1.2, we can get G is supersolvable, the final contradiction.

Case 2 $F \neq U$.

By case 1, H is supersolvable, so H is solvable and $F^*(H) = F(H)$ by Lemma 1.5. Then $G \in F$ by Theorem 2.3.

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量子纠缠首次在电晶体线路中完美实现

量子纠缠是一种量子力学现象,具有量子纠缠现象的成员系统们,即使距离遥远,仍保有特别的关联性,仿若“心有灵犀”一般,也就是当其中一颗电子由于被操作,状态发生变化时,另一颗也会即刻发生相应的状态变化,仿佛两颗电子拥有超光速的秘密通信一般。这种被爱因斯坦时代的科学家们称为“鬼魅似的远距作用”的现象,实际上是打开人类进入量子世界的“钥匙”,其一旦纯化,包括量子密码术、量子信息学及量子计算机,都将应运而生。

然而常见的状况是:受实验条件的限制和不可避免的环境噪声的影响,科学家们制备出来的纠缠态并非都是最大纠缠态,慢慢会退化成为混合态,而使用这种混合纠缠态进行量子通信和量子计算都将会导致信息失真。最近,德国科学家终于首次实现了高度完美化的纠缠态。其类似于光子的纠缠,在光学系统中,光子即使经分光后,仍然表现为“一致行动”。现在,科研人员利用超导体中的电子取代光子来作为电路中的粒子,虽然两个量子点只相距1微米左右,但对于此类实验来说,这个距离大到足以证明纠缠态,物理学家终于在全固体材料中完美演绎了实验,首次确凿地证明从电晶体装置中分离出来的粒子仍可实现量子纠缠。这是量子力学的一次突破性进展。量子纠缠在全固体材料中的完美实现,意味着量子力学真正走进了电子元件中,量子纠缠和全固体材料结合的目的就是实现量子计算以及更加固若金汤的通信。

(据科学网)