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Existence of Periodic Solutions for a Kind of Neutral Nonlinear Differential with Infinity Delay*

具有无限时滞中立型非线性微分方程周期解的存在性

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Abstract The existence and uniqueness of periodic solutions of a kind of nonlinear neutral differential equation with infinity delay is shown by using fixed point principle. In particular, the requirement of Lipschitz condition on the nonlinear function f is essentially dropped, which allows the equation to include a variety of nonlinearities. Meanwhile, an examples are given to illustrate the main results.

Key words differential equation, infinity delay, periodic solution, unique solution, fixed point principle

摘要:基于 Krasnoselskiis不动点定理,给出一类具有无穷时滞中立型微分方程存在唯一周期解的一组充分条件并用例子说明主要结果的可行性.该条件无需非线性项 f 满足 Lipschitz条件,使得方程的应用范围更宽.

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Theory of functional differential equations with delay has undergone a rapid development in the previous fifty years[1~3]. More recently researchers have given special attentions to the study of equations in which the delay argument occurs in the derivative of the state variable as well as in the independent variable, so-called neutral differential equations. In particular, qualitative analysis such as periodicity and stability of solutions of neutral differential equations has been studied extensively by many authors[4~7]. Neutral differential equations have many applications. For example, these equations arise in the study of two or more simple oscillatory systems with some interconnections between them[8,9], and in modeling physical problems such as vibration of masses attached to an elastic bar. On the one hand, note that the following assumptions:

(H_0) The nonlinear function f is Lipschitz with Lipschitz constant L, i.e.,

$$|f(h) - f(t, j)| \leqslant L|h - j|, \text{ for all } h, j \in R;$$

(H_1) There exist two non-negative constants p and q such that

$$|f(h)| \leqslant p|h| + q, \text{ for all } h \in R,$$

have been considered as fundamental conditions for the considered existence of periodic solutions of systems in the literature mentioned above. On the other hand, there are many bounded monotone-nondecreasing function which satisfy neither condition (H_0) nor condition (H_1) in differential systems, such as networks systems. In fact, reference [10, 11] describe several such functions that are not satisfy Lipschitzian.

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Thus, it is necessary to drop conditions (H_0) and (H_1) when one considers periodic solutions of different systems.

This paper is largely motivated by the fact mentioned above and reference [6, 7, 12]. Consider the following nonlinear system:

$$\frac{d}{dt}\left(x(t)-\int_{-\infty}^{t} b(t,s)x(s)\,ds\right) = -a(t)x(t) + \int_{-\infty}^{t} C(t,s,x(s))\,ds + f(t,x_t) + h(t), \qquad (1)$$

where $a(t)$ and $h(t)$ are continuous real valued functions. The functions $b: R\times R\to R$ and $C: R\times R\times R\to R$ are continuous in their respective arguments, $f: R\times E\to R$ is continuous, where E is the space of functions mapping from $(-\infty, 0]$ into R. For every t, the function $x_t\in E$ is defined by $x_t(\theta) = x(t+\theta)$, for $\theta\in(-\infty, 0]$. By using the fixed point principle, the periodic solutions of the scale system (1) without assumptions (H_0) and (H_1) on the nonlinear function f is studied. Some results of the existence and uniqueness of periodic solutions of system are obtained. In particular, the requirement of Lipschitz condition is essentially dropped on the nonlinear function f, which allows system (1) to include a variety of nonlinearities. One also must point out that the idea of dropping the requirement of Lipschitz condition on the nonlinear function f is stimulated by the works of Feng etc[10]. Further study the vector form of system (1) with nonlinear term f without the assumptions (H_0) and (H_1) will be our next stage of the research priorities.

1 Existence of periodic solutions

In system (1), one further assumes that $a(t+T) = a(t)$, $h(t+T) = h(t)$, $b(t+T, s+T) = b(t,s)$, $C(t+T, s+T, \cdot) = C(t,s,\cdot)$, $f(t+T, \varphi) = f(t, \varphi)$, $x_t(s) = x(t+s)$, $s\in(-\infty, 0]$, $t\in R$, and $\int_0^T a(s)\,ds > 0$. Assume that function $C(t,s,x(t))$ satisfies the following hypothesis:

(i) there is the continuous function $M(t,s)$ such that $|C(t,s,x(t))| \leqslant |M(t,s)||x(s)|$;

(ii) function $M(t,s)$ is $M(t+T, s+T) = M(t,s)$ for all $t,s\in R$ and $\int_{-\infty}^{t} |M(t,s)|\,ds$ is bounded, for all $t\in R$.

Assume that $\int_{-\infty}^{t} b(t,s)\,ds$ exists for all $t\in R$ throughout this paper.

Lemma 1 If the continuous function $x(t)$ is periodic in t of period T, $\int_{-\infty}^{t} C(t,s,x(t))\,ds$ is a continuous T-periodic function.

For $T>0$, let B_T be the set of all continuous scalar functions $x(t)$, periodic in t of period T. Then $(B_T, \|\cdot\|)$ is a Banach space with the supremum norm $\|x\| = \sup_{t\in R}|x(t)| = \sup_{t\in[0,T]}|x(t)|$.

To simplify notations, one introduces the following notions.

$$\eta = \max_{t\in[0,T]}\left|\left(1-\exp\left(-\int_{t-T}^{t} a(s)\,ds\right)\right)^{-1}\right|, \quad \bar{a} = \max_{t\in[0,T]}|a(t)|,$$

$$r = \max_{u\in[t-T,T]}\exp\left(-\int_u^t a(s)\,ds\right), \quad H = \max_{t\in[0,T]}|h(t)|,$$

$$\bar{b} = \sup_{t\in R}\int_{-\infty}^{t} b(t,s)\,ds, \quad \bar{M} = \sup_{t\in R}\int_{-\infty}^{t}|M(t,s)|\,ds. \qquad (2)$$

Theorem 1 Suppose that hypothesis (i), (ii), $\bar{b} = \sup_{t\in R}\int_{-\infty}^{t} b(t,s)\,ds < 1$, and (iii) $0 < \bar{f} = \sup_{t\in R,\, x\neq 0}\dfrac{f(t, x+x^*) - f(t, x^*)}{\|x\|} < \infty$ hold. If there is a positive constant J satisfying the inequality

$$\bar{b}J + \eta r T[\bar{a}\bar{b}J + \bar{M}J + \bar{f}J + H] \leqslant J,$$

system (1) has a solution in M, where $M = \{\varphi\in B_T: \|\varphi\|\leqslant J\}$.

Proof First, one proves that if $x(t)\in B_T$, $x(t)$ is a solution of system (1) if and only if

$$x(t) = \int_{-\infty}^{t} b(t,s)x(s)\,ds + \left(1-\exp\left(-\int_{t-T}^{t} a(s)\,ds\right)\right)^{-1}\cdot \int_{t-T}^{t}\left[-a(s)\int_{-\infty}^{s} b(s,\tau)x(\tau)\,d\tau + \int_{-\infty}^{s} C(s,\tau,x(\tau))\,d\tau + f(s,x_s) + h(s)\right]\exp\left(-\int_s^t a(\tau)\,d\tau\right)ds. \qquad (3)$$

Rewrite system (1) into the following form:

$$\frac{d}{dt}\left(x(t)-\int_{-\infty}^{t} b(t,s)x(s)\,ds\right) = -a(t)\left(x(t)-\int_{-\infty}^{t} b(t,s)x(s)\,ds\right) - a(t)\int_{-\infty}^{t} b(t,s)x(s)\,ds + \int_{-\infty}^{t} C(t,s,x(s))\,ds + f(t,x_t) + h(t).$$

Multiply both sides of system (1) with $\exp\left(\int_0^t a(s)\,ds\right)$ and integrate from $t-T$ to t to obtain

$$\int_{t-T}^{t}\left[x(\tau) - \int_{-\infty}^{s} b(s,\tau)x(\tau)\,d\tau\right]'ds = \int_{t-T}^{t}\left[-a(s)\int_{-\infty}^{s} b(s,\tau)x(\tau)\,d\tau + \int_{-\infty}^{s} C(s,\tau,x(\tau))\,d\tau + f(s,\right.$$

$x_s) + h(s)]\cdot \exp(\int_0^s a(f)\,df)\,ds.$

As a result, one arrives at

$$[x(t) - \int_{-\infty}^{t} b(t,s)x(s)\,ds]\exp(\int_0^t a(s)\,ds) - [x(t-T) - \int_{-\infty}^{t-T} b(t-T,s)x(s)\,ds]\cdot \exp(\int_0^{t-T} a(s)\,ds) = \int_{t-T}^{t} [-a(s)\int_{-\infty}^{s} b(s,f)x(f)\,df + \int_{-\infty}^{s} C(s,f,x(f))\,df + f(s,x_s) + h(s)]\cdot \exp(\int_0^s a(f)\,df)\,ds.$$

Dividing both sides of the above equation by $\exp(\int_0^t a(s)\,ds)$ and noting the fact that

$$x(t) = x(t-T) = x(t+T), b(t-T,s-T) = b(t,s),$$

one can deduce

$$[x(t) - \int_{-\infty}^{t} b(t,s)x(s)\,ds] = [1 - \exp(\int_t^{t-T} a(s)\,ds)]^{-1}[\int_{t-T}^{t}(-a(s)\int_{-\infty}^{s} b(s,f)x(f)\,df + \int_{-\infty}^{s} C(s,f,x(f))\,df + f(s,x_s) + h(s))\exp(\int_t^s a(f)\,df)\,ds].$$

Define a mapping

$$L(x(t)) = F(x(t)) + G(x(t)), \quad (4)$$

where

$$F(x(t)) = \int_{-\infty}^{t} b(t,s)x(s)\,ds, \quad (5)$$

$$G(x(t)) = [1 - \exp(\int_t^{t-T} a(s)\,ds)]^{-1}[\int_{t-T}^{t}(-a(s)\int_{-\infty}^{s} b(s,f)x(f)\,df + \int_{-\infty}^{s} C(s,f,x(f))\,df + f(s,u_s) + h(s))\exp(\int_t^s a(f)\,df)\,ds]. \quad (6)$$

It is clear that $L: B_T \to B_T$ by the way it was constructed.

Secondly, to show that G is continuous and the image of G is contained in a compact set. Let $h, j \in B_T$. Given $X > 0$, take $W = X/N$ with $N = Z_r T(\bar{a}\bar{b} + \bar{M} + \bar{f})$, where $\bar{a}, \bar{b}, \bar{M}$ and $\bar{f}$ are given in Theorem 1. For $\|h - j\| < W$, one deduces

$$\|Gh - Gj\| = [1 - \exp(\int_t^{t-T} a(s)\,ds)]^{-1}|\int_{t-T}^{t}(|a(u)|\int_{-\infty}^{u} |b(u,s)|\cdot |h(s) - j(s)|\,ds + \int_{-\infty}^{u} |M(u,s)||h(s) - j(s)|\,ds + |f(u,h_u) - f(u,j_u)|)\cdot \exp(\int_t^u a(s)\,ds)\,du| \leqslant Z_r\int_0^T(\bar{a}\bar{b} + \bar{M} + \bar{f})\|h - j\|\,ds \leqslant Z_r T(\bar{a}\bar{b} + \bar{M} + \bar{f})\|h - j\| < X. \quad (7)$$

This proves that G is continuous. In order to show that the image of G is contained in a compact set, consider $D = \{h \in B_T: \|h\| \leqslant R\}$, where R is a fixed positive constant. Let $h_n \in D$ where n is a positive integer. Obviously, $\|Gh_n\| \leqslant L$, where L is positive constant. Next, we calculate $(Gh_n)'(t)$ and show that Gh_n is uniformly bounded. By making use of $\int_0^T a(s)\,ds > 0$ and periodicity of function b, c and f, one obtains by taking the derivative in formula (6) that

$$(Gh_n)'(t) = -a(t)G(h_n(t)) - a(t)\int_{-\infty}^{t} b(t,s)h_n(s)\,ds + \int_{-\infty}^{t} C(t,s,h_n(s))\,ds + f(t,h_{nt}) + h(t). \quad (8)$$

Thus, the above expression yields $\|(Gh_n)'\| \leqslant F$, for some positive constant F, which implies that the sequence Gh_n is uniformly bounded and equi-continuous. Hence by Ascoli-Arzela's theorem, $G(D)$ is compact.

Thirdly, it is easy to show that F is a contraction. In fact, for any $h, j \in B_T$, one has

$$\|F(h) - F(j)\| = \sup_{t \in R}|F(h) - F(j)| \leqslant \bar{b}\sup_{t \in [0,T]}|h(t) - j(t)| \leqslant \bar{b}\|h - j\|.$$

Hence F defines a contraction.

Finally, define $M = \{h \in B_T: \|h\| \leqslant J\}$. From above discussion, one knows that $F: B_T \to B_T$. Further, to show that $h, j \in M$ implies $\|Gh + Fj\| \leqslant J$. Let $h, j \in M$, with $\|h\|, \|j\| \leqslant J$, then

$$\|Gh + Fj\| \leqslant \bar{b}\|j\| + Z_r\int_0^T[|a(s)|\bar{b}\|h\| + \bar{M}\|h\| + \bar{f}\|h\| + H]\,ds \leqslant \bar{b}\|j\| + Z_r T[\bar{a}\bar{b}\|h\| + \bar{M}\|h\| + \bar{f}\|h\| + H] \leqslant \bar{b}J + Z_r T[\bar{a}\bar{b}J + \bar{M}J + \bar{f}J + H] \leqslant J. \quad (9)$$

Then all the conditions of Krasnoselskii's theorem[13] are satisfied. Thus there exists a fixed point z in M such that $z = Fz + Gz$. Hence equation (1) has a T-periodic solution.

Corollary 1 Let $M = \{h \in B_T: \|h\| \leqslant J\}$. Suppose all conditions of Theorem 1 hold. If

$$\bar{b} + Z_r T[\bar{a}\bar{b}J + \bar{c}J + \bar{f}J] < 1,$$

system (1) has a unique T-periodic solution in M.

Proof Let the mapping L be given by formula (4)~(6). For $h, j \in M$, one has

$$\|Lh - Lj\| \leqslant (\bar{b} + Z_r T(\bar{a}\bar{b} + \bar{M} + \bar{f}))\|h - j\|$$

(10)

in view of formula (4)~ (6). This completes the proof by invoking the contraction mapping principle.

Remark 1 Indeed condition (iii) is weaker than the Lipschitz condition (H_0). If f is Lipschitz function, $\bar{f}$ can be replaced by the respective Lipschitz constants. Therefore, condition (iii) includes the case that f is Lipschitz function. Thus, this paper essentially drop the requirement of Lipschitz condition on the nonlinear function f in the above-mentioned literature, which allows system (1) to include a variety of nonlinearities, and improve and extend interrelated results.

2 An example

Example 1 For small positive X_1 and X_2, consider the following equation

$$\frac{d}{dt}(x(t) - X_1\int_{-\infty}^{t} e^{-(t-s)}x(s)ds) = -(2+\cos t)x(t) + X_2\int_{-\infty}^{t} e^{-(t-s)}x(s)ds + \int_{-\infty}^{t} W(s)e^{-(x(t-\sin t)-s)}ds + \sin t - 1, \quad (11)$$

where $b(t,s) = X_1 e^{-(t-s)}$, $C(t,s,x(s)) = X_2 e^{-(t-s)}x(s)$, $M(t,s) = X_2 e^{-(t-s)}$, $f = \int_{-\infty}^{t} W(s)e^{-(x(t-\sin t)-s)}ds$, pulse function W equals one if a pulse arrives at time t or zero, if no pulse arrives, $a(t) = 2 + \cos t$, $h(t) = \sin t - 1$.

Obviously, function f does not satisfy the conditions (H_0) and (H_1). Further, $\bar{f} = 1$, $H = 2$, $T = 2\pi$, $\bar{a} = 3$, $\bar{b} = X_1$, $\bar{M} = X_2$, $Z = (1 - e^{-4\pi})^{-1}$, $r \leqslant 1$. Let a constant $J_1 = -4 c r e^{4\pi} / [1 - X_1 + e^{4\pi}(X_1 + 6 X_1 \pi r + 2(1 + X_2)\pi r - 1)]$. For small X_1 and X_2, there is always a positive $J \geqslant J_1$ such that

$$\bar{b}J + Z r T[\bar{a}\bar{b}J + \bar{M}J + \bar{f}J + H] \leqslant J$$

hold. Hence, equation (11) has a 2π-periodic solution by Theorem 1.

Choose a constant $J_2 = \frac{(1 - e^{-4\pi})(1 - X_1)}{2\pi r(1 + 3X_1 + X_2)}$. For small X_1 and X_2, choose positive J with $J_2 \geqslant J \geqslant J_1$ such that

$$\bar{b} + Z r T[\bar{a}\bar{b}J + \bar{M}J + \bar{f}J] < 1$$

hold. Hence, equation (11) has an unique 2π-periodic solution by Corollary 1.

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