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Existence of Periodic Solutions for a Kind of High Order Liénard Equations with Deviating Argument*

一类具偏差变元的高阶 Liénard型方程周期解的存在性

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Abstract By using coincidence degree theory, the sufficient conditions are obtained for existence of at least one periodic solution for a kind of high order Lienard equations with deviating argument.

Key words differential equation, Liénard equations, periodic solution, existence, coincidence degree

摘要: 利用重合度理论,得到一类具偏差变元的高阶 Liénard型方程至少存在一个周期解的充分条件.

关键词: 微分方程 Liénard型方程 周期解 存在性 重合度
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The existence of periodic solutions for Liénard equations with deviating argument has been studied extensively[1-4]. The purpose of the article is to investigate the Liénard equations with deviating arguments of the form

$$x^{(n)} + f(t,x(t),x(t-\tau_0(t)))x'(t) + g(t, x(t),x(t-\tau_1(t)),\cdots,x(t-\tau_m(t))) = p(t), \quad (1)$$

where $f\in C(R^3,R)$, $g\in C(R^{m+2},R)$ are continuous functions and $f(t+T,x,y) = f(t,x,y)$, $\forall (x,y)\in R^2$, $g(t+T,x_0,x_1,\cdots,x_m) = g(t,x_0,x_1,\cdots,x_m)$, $\forall (x_0,x_1,\cdots,x_m)\in R^{m+1}$, $P\in C(R,R)$, $\tau_i\in C(R,R)(i=0,1,2,\cdots,m)$ with $p(t+T) = p(t)$, $\tau_i(t+T) = \tau_i(t)$ and $T\in(0,2\pi]$.

In reference [5], Lu discussed the existence of periodic solutions for a kind of second order differential equations with deviating arguments

$$x''(t) + f(t,x(t),x(t-\tau_0(t)))x'(t) + \beta(t)g(x(t-\tau_1(t))) = p(t).$$

In reference [6], Wang studied a kind of high order Liénard equations of the type

$$x^{(n)}(t) + f(x(t))x'(t) + g(x-\tau(t,x(t))) = p(t).$$

In reference [7], Pan studied the existence of periodic solutions of the following differential equations

$$x^{(n)}(t) = \sum_{i=1}^{n-1} b_i x^{(i)}(t) + f(t,x(t),x(t-\tau_1(t)), \cdots,x(t-\tau_m(t))) + p(t).$$

In this paper, by using Mawhin's continuation theorem, we establish a theorem on the existence of periodic solutions of equation (1). Our method is different from reference [7], and the results of this paper are new and complement previously ones[1-4]. In addition, we give an example to demonstrate our new results.

1 Preliminaries

For ease of exposition, throughout this paper we adopt the following notations: Let $Y = \{x\in C(R,R)| x(t+T) = x(t)\}$ with the norm $|x|_\infty = \max_{t\in[0,T]}\{|x(t)|\}$ and $X = \{x\in C^{n-1}(R,R)| x(t+T) = x(t)\}$ with norm $\|x\| = \max\{|x|_\infty, |x'|_\infty, \cdots, |x^{(n-1)}|_\infty\}$ be two Banach spaces, $L: D(L)\subset X\to Y$ be a Fredholm operator of index zero. $P: X\to X$, $Q: Y\to Y$ be projectors such that

$$\mathrm{Im}P = \ker L, \ker Q = \mathrm{Im}L, X = \ker L\oplus\ker P, Y = \mathrm{Im}L\oplus\mathrm{Im}Q.$$

It follows that $L|_{D(L)\cap\ker P}: D(L)\cap\ker P\to\mathrm{Im}L$ is invertible, we denote the inverse of the map by K_p. Let Ω be an open bounded subset of X, $D(L)\cap\bar{\Omega}\neq\emptyset$, the map $N: X\to Y$ will be called L-compact in $\bar{\Omega}$,

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if $QN(\bar{\Omega})$ is bounded and $K_p(I-Q)N: \bar{\Omega} \to X$ is compact. We also define the operators L and N as follows:

$L: D(L) \subset X \to Y, Lx = x^{(n)}, D(L) = \{x | x \in C^n(R,R), x(t+T) = x(t)\},$ (2)

$N: X \to Y, Nx = -f(t,x(t),x(t-\tau_0(t))) \cdot x'(t) - g(t,x(t),x(t-\tau_1(t)),\cdots,x(t-\tau_m(t))) + p(t).$ (3)

It is easy to see equation(1) can be converted to the abstract equation $Lx = Nx$. Moreover, from the definition of L, we can see $\ker L = R$, $\dim(\ker L) = 1$, $\mathrm{Im}L = \{y | y \in Y, \int_0^T y(s)\mathrm{d}s = 0\}$ is subset, and $\dim(Y/\mathrm{Im}L) = 1$, we have $\mathrm{codim}(\mathrm{Im}L) = \dim(\ker L)$, so L is a Fredholm operator with index zero. Let $P: X \to \ker L$, $Px = x(0)$, $Q: Y \to Y/\mathrm{Im}L$, $Qy = \frac{1}{T}\int_0^T y(t)\mathrm{d}t$ and let $L|_{D(L)\cap \ker P}: D(L) \cap \ker P \to \mathrm{Im}L$. Then $L|_{D(L)\cap \ker P}$ has a unique continuous inverse K_p. One can easily find that N is L-compact in $\bar{\Omega}$, where Ω is an open bounded subset of X.

Lemma 1[8] Suppose $x \in C^n(R,R)$ and $x(t+T) = x(t)$ $(0 < T \leqslant 2\pi)$, then

$$\left(\int_0^T |x'(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \leqslant \left(\int_0^T |x''(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}} \leqslant \cdots \leqslant \left(\int_0^T |x^{(n)}(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}. \quad (4)$$

Lemma 2[9] Let $\tau \in [0, +\infty)$ be constants, $s \in C(R,R)$ with $s(t+T) = s(t)$, and $s(t) \in [-\tau, \tau]$, $\forall t \in [0,T]$. Then $\forall x \in C^1(R,R)$ with $x(t+T) = x(t)$, we have

$$\int_0^T |x(t) - x(t-s(t))|^2 \mathrm{d}t \leqslant 2\tau^2 \int_0^T |x'(t)|^2 \mathrm{d}t. \quad (5)$$

Lemma 3[10] Let L be a Fredholm operator of index zero and let N be L-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied:

(i) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0,1)$;

(ii) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \ker L$;

(iii) $\deg\{QN, \Omega \cap \ker L, 0\} \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap D(L)$.

2 Main results

Theorem Assume the following conditions hold:

(H_1) $\sup_{(t,x,y)\in R^3} |f(t,x,y)| = A$.

(H_2) There is a constant $c > 0$, such that

$$|g(t,x_0,x_1,\cdots,x_m) + f(t,x_0,x_{m+1})x'| > |p(t)|_\infty, \quad (6)$$

$\forall t \in R, |x_i| > c, i = 0,1,\cdots,m,m+1.$

(H_3) The function g has the decomposition

$$g(t,x_0,x_1,\cdots,x_m) = K(t,x_0) + \sum_{i=1}^m h_i(t,x_i), \quad (7)$$

such that

$$|K(t,x)| \leqslant \alpha_1 + \alpha_2|x|, \quad (8)$$

$$|h_i(t,x) - h_i(t,y)| \leqslant \beta_i|x-y|, i = 1,\cdots,m, \quad (9)$$

and

$$\lim_{|x|\to\infty} \left|\frac{h_i(t,x)}{x}\right| \leqslant \gamma_i, i = 1,\cdots,m, \quad (10)$$

where $\alpha_1, \alpha_2, \beta_i, \gamma_i > 0$.

Then equation(1) has at least one T-periodic solution provided one of the following conditions hold $(0 < T \leqslant 2\pi)$:

(A_1) $2^{\frac{1}{2}}\sum_{i=1}^m |\tau_i(t)|_\infty \beta_i + \alpha_2 T + T\sum_{i=1}^m \gamma_i + A < B$, $n = 2k, k \in Z^+$,

where $B = \min\{1, \frac{1}{T}\}$.

(A_2) $2^{\frac{1}{2}}\sum_{i=1}^m |\tau_i(t)|_\infty \beta_i + \alpha_2 T + T\sum_{i=1}^m \gamma_i + A < 1$. $n = 2k-1, k \in Z^+$.

Proof Consider the equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, where L and N are defined by formula(2) and formula(3). Let $\Omega_1 = \{x \in D(L)/\ker L, Lx = \lambda Nx$ for some $\lambda \in (0,1)\}$ for $x \in \Omega_1$. We have

$$x^{(n)} + \lambda[f(t,x(t),x(t-\tau_0(t)))x'(t) + g(t,x(t),x(t-\tau_1(t)),\cdots,x(t-\tau_m(t)))] = \lambda p(t). \quad (11)$$

Integrating them on $[0,T]$, we have

$$\int_0^T [f(t,x(t),x(t-\tau_0(t)))x'(t) + g(t,x(t),x(t-\tau_1(t)),\cdots,x(t-\tau_m(t)))]\mathrm{d}t = \int_0^T p(t)\mathrm{d}t. \quad (12)$$

We can prove that there is $t_1 \in [0,T]$ such that $|x(t_1)| < c$. Indeed, from formula(12), there is $t_0 \in [0,T]$ such that

$$f(t_0,x(t_0),x(t_0-\tau_0(t_0)))x'(t_0) + g(t_0,x(t_0),x(t_0-\tau_1(t_0)),\cdots,x(t_0-\tau_m(t_0))) = \frac{1}{T}\int_0^T p(t)\mathrm{d}t.$$

If $|x(t_0)| \leqslant c$, then taking $t_1 = t_0$ such that $|x(t_1)| \leqslant c$.

If $|x(t_0)| > c$, it follows from assumptions (H_2) that there is some $i \in \{1,\cdots,m,m+1\}$ such that $|x(t_0 - \tau_i(t_0))| \leqslant c$. Since $x(t)$ is continuous for $t \in R$ and $x(t+T) = x(t)$, so there must be an integer k and a point $t_1 \in [0,T]$ such that $t_0 - \tau_i(t_0) = kT + t_1$. So $|x(t_1)| = |x(t_0 - \tau_i(t_0))| < c$, which implies

$$|x(t)|_\infty \leqslant c + \int_0^T |x'(t)|\,dt. \qquad (13)$$

Suppose $n = 2k$ for a positive integer. Then, multiplying both sides of formula (11) by $x(t)$, and integrating them on $[0, T]$, we have for $\lambda \in (0, 1)$.

$$(-1)^k \int_0^T |x^{(k)}(t)|^2 dt = -\lambda \int_0^T g(t, x(t), x(t-\tau_1(t)), \cdots, x(t-\tau_m(t)))x(t)\,dt - \lambda \int_0^T f(t, x(t), x(t-\tau_0(t)))x'(t)x(t)\,dt + \lambda \int_0^T x(t)p(t)\,dt = -\lambda \int_0^T f(t, x(t), x(t-\tau_0(t)))x'(t)x(t)\,dt - \lambda \int_0^T K(t, x(t))x(t)\,dt - \lambda \int_0^T \sum_{i=1}^m h_i(t, x(t-\tau_i(t)))x(t)\,dt + \lambda \int_0^T x(t)p(t)\,dt.$$

Thus, we have

$$\int_0^T |x^{(k)}(t)|^2 dt \leqslant A \int_0^T |x'(t)||x(t)|\,dt + \int_0^T |K(t, x(t))||x(t)|\,dt + \int_0^T |p(t)||x(t)|\,dt + \sum_{i=1}^m \int_0^T |h_i(t, x(t-\tau_i(t))) - h_i(t, x(t))||x(t)|\,dt + \sum_{i=1}^m \int_0^T |h_i(t, x(t))||x(t)|\,dt. \qquad (14)$$

Choosing a constant $\varepsilon > 0$ such that $AT + 2^{\frac{1}{2}} T \sum_{i=1}^m |\tau_i(t)|_\infty \alpha_i + \beta_2 T^2 + T^2 \sum_{i=1}^m (\gamma_i + \varepsilon) < 1$. For the above constant $\varepsilon > 0$, from formula (10) we see that there is a constant $\rho > 0$ such that

$$|h_i(t, x)| < (\gamma_i + \varepsilon)|x|, \text{ for } |x| > \rho, t \in [0, T]. \qquad (15)$$

Denote

$$\Delta_1 = \{t \in [0, T] : |x(t)| \leqslant \rho\}, \Delta_2 = \{t \in [0, T] : |x(t)| > \rho\}, \qquad (16)$$

and then it follows from formula (13), (15), (16) and Schwarz inequality and Lemma 1 that

$$\int_0^T |h_i(t, x(t))||x(t)|\,dt = \int_{\Delta_1} |h_i(t, x(t))||x(t)|\,dt + \int_{\Delta_2} |h_i(t, x(t))||x(t)|\,dt \leqslant h_{i\rho} \int_0^T |x(t)|\,dt + (\gamma_i + \varepsilon)|x|_\infty \int_0^T |x(t)|^2 dt \leqslant h_{i\rho}|x(t)|_\infty T + (\gamma_i + \varepsilon)|x(t)|_\infty^2 T, \qquad (17)$$

where $h_{i\rho} = \max_{t \in [0,T], |x| \leqslant \rho} |h_i(t, x)|$. Furthermore, from formula (9), by using Schwarz inequality, Lemma 1 and Lemma 2, we obtain

$$\int_0^T |h_i(t, x(t-\tau_i(t))) - h_i(t, x(t))||x(t)|\,dt \leqslant |x(t)|_\infty \alpha_i \int_0^T |x(t) - x(t-\tau_i(t))|\,dt \leqslant (2T)^{\frac{1}{2}} |x(t)|_\infty \alpha_i |\tau_i(t)|_\infty \left(\int_0^T |x^{(k)}(t)|^2 dt\right)^{\frac{1}{2}}. \qquad (18)$$

From formula (8), (13), we get

$$\int_0^T |K(t, x(t))||x(t)|\,dt \leqslant \int_0^T [\beta_1 + \beta_2 |x(t)|]|x(t)|\,dt \leqslant \beta_1 |x(t)|_\infty T + \beta_2 |x(t)|_\infty^2 T. \qquad (19)$$

$$\int_0^T |p(t)||x(t)|\,dt \leqslant T|p(t)|_\infty |x(t)|_\infty. \qquad (20)$$

Substituting formula (17)~(20) into formula (14), we have

$$\int_0^T |x^{(k)}(t)|^2 dt \leqslant \left[AT + 2^{\frac{1}{2}} T \sum_{i=1}^m |\tau_i(t)|_\infty \alpha_i + \beta_2 T^2 + T^2 \sum_{i=1}^m (\gamma_i + \varepsilon)\right] \int_0^T |x^{(k)}(t)|^2 dt + \left[AT^{\frac{1}{2}} c + (2T)^{\frac{1}{2}} c \sum_{i=1}^m |\tau_i(t)|_\infty \alpha_i + \beta_1 T^{\frac{3}{2}} + T^{\frac{3}{2}} |p(t)|_\infty + T^{\frac{3}{2}} \sum_{i=1}^m h_{i\rho} + 2c\beta_2 T^{\frac{3}{2}} + 2cT^{\frac{3}{2}} \sum_{i=1}^m (\gamma_i + \varepsilon)\right] \left(\int_0^T |x^{(k)}(t)|^2 dt\right)^{\frac{1}{2}} + \left[\beta_1 Tc + Tc|p(t)|_\infty + Tc \sum_{i=1}^m h_{i\rho} + \beta_2 Tc^2 + c^2 T \sum_{i=1}^m (\gamma_i + \varepsilon)\right]. \qquad (21)$$

From (A_1), we obtain that there is a constant $M_1 > 0$ such that $\int_0^T |x^{(k)}(t)|^2 dt \leqslant M_1$.

Suppose $n = 2k - 1$ for a positive integer. Then, multiplying both sides of formula (11) by $x'(t)$, and integrating them on $[0, T]$, we have for $\lambda \in (0, 1)$.

$$(-1)^{k-1} \int_0^T |x^{(k)}(t)|^2 dt = -\lambda \int_0^T g(t, x(t), x(t-\tau_1(t)), \cdots, x(t-\tau_m(t)))x'(t)\,dt - \lambda \int_0^T f(t, x(t), x(t-\tau_0(t)))(x'(t))^2 dt + \lambda \int_0^T x'(t)p(t)\,dt = -\lambda \int_0^T f(t, x(t), x(t-\tau_0(t)))(x'(t))^2 dt - \lambda \int_0^T K(t, x(t))x'(t)\,dt - \lambda \int_0^T \sum_{i=1}^m h_i(t, x(t-\tau_i(t)))x'(t)\,dt + \lambda \int_0^T x'(t)p(t)\,dt.$$

Thus, we have

$$\int_0^T |x^{(k)}(t)|^2 dt \leqslant A \int_0^T |x'(t)|^2 dt + \int_0^T |K(t, x(t))||x'(t)|\,dt + \int_0^T |p(t)||x'(t)|\,dt + \sum_{i=1}^m \int_0^T |h_i(t, x(t-\tau_i(t))) - h_i(t, x(t))||x'(t)|\,dt + \sum_{i=1}^m \int_0^T |h_i(t, x(t))||x'(t)|\,dt. \qquad (22)$$

Choosing a constant $\varepsilon > 0$ such that $2^{\frac{1}{2}} \sum_{i=1}^m |\tau_i(t)|_\infty \alpha_i + \beta_2 T + T \sum_{i=1}^m (\gamma_i + \varepsilon) + A < 1$. And then it follows from formula (13), (15), (16) and Schwarz inequality and Lemma 1 that

$$\int_0^T|h_i(t,x(t))||x'(t)|\,dt = \int_{\Delta_1}|h_i(t,x(t))||x'(t)|\,dt + \int_{\Delta_2}|h_i(t,x(t))||x'(t)|\,dt \leqslant h_{iW}\int_0^T|x'(t)|\,dt + (V_i + X_i)(c + \int_0^T|x'(t)|\,dt)(\int_0^T|x'(t)|\,dt) \leqslant [h_{iW} + (V_i + X_i)c](\int_0^T|x^{(k)}(t)|^2dt)^{\frac{1}{2}}T^{\frac{1}{2}} + (V_i + X_i)T\int_0^T|x^{(k)}(t)|^2dt, \quad (23)$$

where $h_{iW} = \max_{t\in[0,T],|x|\leqslant W}|h_i(t,x)|$. Furthermore, from formula (9), by using Schwarz inequality and Lemma 1 and Lemma 2, we obtain

$$\int_0^T|h_i(t,x(t-f_i(t)))-h_i(t,x(t))|\cdot|x'(t)|\,dt \leqslant T_i\int_0^T|x(t)-x(t-f_i(t))|\cdot|x'(t)|\,dt \leqslant 2^{\frac{1}{2}}T_i|f_i(t)|_\infty\int_0^T|x^{(k)}(t)|^2dt. \quad (24)$$

From formula (8), (13), by Lemma 2, we get

$$\int_0^T|K(t,x(t))||x'(t)|\,dt \leqslant (U_1T^{\frac{1}{2}} + U_2cT^{\frac{1}{2}})(\int_0^T|x^{(k)}(t)|^2dt)^{\frac{1}{2}} + U_2T\int_0^T|x^{(k)}(t)|^2dt. \quad (25)$$

By Schwarz inequality and Lemma 1, we have

$$\int_0^T|p(t)||x'(t)|\,dt \leqslant |p(t)|_\infty\int_0^T|x'(t)|\,dt \leqslant |p(t)|_\infty T^{\frac{1}{2}}(\int_0^T|x^{(k)}(t)|^2dt)^{\frac{1}{2}}. \quad (26)$$

Substituting formula (23)~(26) into formula (22), we have

$$\int_0^T|x^{(k)}(t)|^2dt \leqslant [U_1 + U_2c + \sum_{i=1}^m(h_{iW} + V_ic + X_ic) + |p(t)|_\infty]T^{\frac{1}{2}}(\int_0^T|x^{(k)}(t)|^2dt)^{\frac{1}{2}} + [2^{\frac{1}{2}}\sum_{i=1}^m T_i|f_i(t)|_\infty + U_2T + \sum_{i=1}^m(V_i + X_i)T + A\int_0^T|x^{(k)}(t)|^2dt. \quad (27)$$

From (A2), we obtain that there is a constant $M_2 > 0$ such that $\int_0^T|x^{(k)}(t)|^2dt \leqslant M_2$. Let $M_3 = \max\{M_1, M_2\}$. Thus, $\int_0^T|x^{(k)}(t)|^2dt \leqslant M_3$. By Lemma 1, we have

$$\int_0^T|x'(t)|^2dt \leqslant M_3. \quad (28)$$

On the other hand, multiplying both sides of formula (11) by $x^{(n)}(t)$, we have

$$\int_0^T|x^{(n)}(t)|^2dt \leqslant \int_0^T|f(t,x(t),x(t-f_0(t)))||x'(t)||x^{(n)}(t)|\,dt + \int_0^T|K(t,x(t))||x^{(n)}(t)|\,dt + \int_0^T|p(t)||x^{(n)}(t)|\,dt + \sum_{i=1}^m\int_0^T|h_i(t,x(t-f_i(t)))-h_i(t,x(t))||x^{(n)}(t)|\,dt + \sum_{i=1}^m\int_0^T|h_i(t,x(t))||x^{(n)}(t)|\,dt. \quad (29)$$

Since

$$\int_0^T|h_i(t,x(t))||x^{(n)}(t)|\,dt \leqslant [h_{iW} + (V_i + X_i)c](\int_0^T|x^{(n)}(t)|^2dt)^{\frac{1}{2}}T^{\frac{1}{2}} + (V_i + X_i)T\int_0^T|x^{(n)}(t)|^2dt, \quad (30)$$

by using Schwarz inequality and Lemma 1 and Lemma 2, it follows from formula (9) that

$$\int_0^T|h_i(t,x(t-f_i(t)))-h_i(t,x(t))|\cdot|x^{(n)}(t)|\,dt \leqslant T_i\int_0^T|x(t)-x(t-f_i(t))||x^{(n)}(t)|\,dt \leqslant 2^{\frac{1}{2}}T_i|f_i(t)|_\infty\int_0^T|x^{(n)}(t)|^2dt. \quad (31)$$

From formula (8), (13) and Lemma 1, we get

$$\int_0^T|K(t,x(t))||x^{(n)}(t)|\,dt \leqslant \int_0^T[U_1 + U_2|x(t)|]|x^{(n)}(t)|\,dt \leqslant (U_1 + U_2c)T^{\frac{1}{2}}(\int_0^T|x^{(n)}(t)|^2dt)^{\frac{1}{2}} + U_2T\int_0^T|x^{(n)}(t)|^2dt. \quad (32)$$

By Schwarz inequality and Lemma 1, we have

$$\int_0^T|p(t)||x^{(n)}(t)|\,dt \leqslant |p(t)|_\infty\int_0^T|x^{(n)}(t)|\,dt \leqslant |p(t)|_\infty T^{\frac{1}{2}}(\int_0^T|x^{(n)}(t)|^2dt)^{\frac{1}{2}}. \quad (33)$$

Substituting formula (30)~(33) into formula (29), we have

$$\int_0^T|x^{(n)}(t)|^2dt \leqslant [U_1 + U_2c + \sum_{i=1}^m(h_{iW} + V_ic + X_ic) + |p(t)|_\infty]T^{\frac{1}{2}}(\int_0^T|x^{(n)}(t)|^2dt)^{\frac{1}{2}} + [2^{\frac{1}{2}}\sum_{i=1}^m T_i|f_i(t)|_\infty + U_2T + \sum_{i=1}^m(V_i + X_i)T + A\int_0^T|x^{(n)}(t)|^2dt. \quad (34)$$

From (A2), we obtain that there is a constant $M > 0$ such that $\int_0^T|x^{(n)}(t)|^2dt \leqslant M$. We claim that $|x^{(i)}(t)| \leqslant T^{n-i-1}\int_0^T|x^{(n)}(t)|\,dt, i = 1,2,\cdots,n-1$. In fact, noting that $x^{(n-2)}(0) = x^{(n-2)}(T)$, there must be a constant $Y_1 \in [0,T]$ such that $x^{(n-1)}(Y_1) = 0$, we obtain

$$|x^{(n-1)}(t)| = |x^{(n-1)}(Y_1) + \int_{Y_1}^t x^{(n)}(s)\,ds| \leqslant |x^{(n-1)}(Y_1)| + \int_0^T|x^{(n)}(t)|\,dt = \int_0^T|x^{(n)}(t)|\,dt. \quad (35)$$

Similarly, since $x^{(n-3)}(0) = x^{(n-3)}(T)$, there must be a constant $Y_2 \in [0,T]$ such that $x^{(n-2)}(Y_2) = 0$, from formula (35) we get $|x^{(n-2)}(t)| = |x^{(n-2)}(Y_2) +$

$\int_{Y_2}^{t} x^{(n-1)}(s)\,ds| \leqslant T\int_0^T |x^{(n)}(t)|\,dt$. By induction, we have $|x^{(i)}(t)| \leqslant T^{n-i-1}\int_0^T |x^{(n)}(t)|\,dt, i = 1,2,\cdots,n-1$. Furthermore, we have $|x^{(i)}(t)|_\infty \leqslant T^{n-i-1}\int_0^T |x^{(n)}(t)|\,dt \leqslant T^{n-i-\frac{1}{2}}M^{\frac{1}{2}}, i = 1,2,\cdots,n-1$.

From formula (13) and formula (28), we obtain $|x(t)|_\infty \leqslant c + T^{\frac{1}{2}}(\int_0^T |x'(t)|^2 dt)^{\frac{1}{2}} \leqslant c + T^{\frac{1}{2}}M_3^{\frac{1}{2}}$. It follows that there is a constant $B > 0$ such that $\|x\| \leqslant B$. Thus Ω_1 is bounded.

Let $\Omega_2 = \{x \in \ker L, QNx = 0\}$. Suppose $x \in \Omega_2$, then $|x(t)| = d \in R$ and satisfies

$$QNx = \frac{1}{T}\int_0^T [-g(t,d,d,\cdots,d) + p(t)]dt = 0. \qquad (36)$$

From formula (36) and assumption (H_2), we have $d \leqslant c$, which implies Ω_2 is bounded. Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \overline{\Omega} \cup \overline{\Omega_2}$. We can easily see that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. Then by the above argument we have (i) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0,1)$; (ii) $QNx \neq 0, \forall x \in \partial\Omega \cap \ker L$.

At last we will prove (iii) of Lemma 3 is satisfied. We take $H(x,\_): \overline{\Omega} \times [0,1] \to X$, $H(x,\_) = \_x + \frac{1-\_}{T}\int_0^T [-g(t,x(t),x(t-f_1(t)),\cdots,x(t-f_m(t))) + p(t)]dt$. From assumption ($H_2$), we can easily obtain $H(x,\_) \neq 0, \forall (x,\_) \in \partial\Omega \cap \ker L \times [0,1]$, which results in $\deg\{QNx, \Omega \cap \ker L, 0\} = \deg\{H(x,0), \Omega \cap \ker L, 0\} = \deg\{H(x,1), \Omega \cap \ker L, 0\} \neq 0$.

Hence, by using Lemma 3, we know that equation (1) has at least one T-periodic solution.

3 Example

Consider the following equation $x^{(5)}(t) + \frac{1}{6}(1 + \sin x(t) + \sin x(t - \cos t))x'(t) + (\frac{1}{30}\cos t)x(t) + (\frac{1}{28}\sin t)x(t - \frac{1}{100}\cos t) = \cos t$. Where $n = 5$, $A = \frac{1}{2}$, $g(t,x) = (\frac{1}{30}\cos t)x$, $h_1(t,y) = (\frac{1}{28}\sin t)y$, $p(t) = \cos t$, $f_1(t) = \frac{1}{100}\cos t$. Thus, $f_1'(t) = \frac{1}{100}\cos t$. $T = 2\pi$, $U_2 = \frac{1}{30}$, $T_1 = \frac{1}{28}$, $V_1 = \frac{1}{28}$, $|f_1'(t)|_\infty = \frac{1}{100}$. Obviously assumptions (H_1) ~ (H_3) hold and $2^{\frac{1}{2}}T_1|f_1'(t)|_\infty + U_2 T + V_1 T + A = 2^{\frac{1}{2}} \times \frac{1}{28} \times \frac{1}{100} + \frac{1}{30} \times 2\pi + \frac{1}{28} \times 2\pi + \frac{1}{2} < 1$. By Theorem, we know that this equation has at least one 2π-periodic solution.

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