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弱拟三角 Hopf 代数的研究*
On Weakly-Quasitriangular Hopf Algebra

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摘要:证明了当 A,B 为弱拟三角 Hopf 代数时,张量代数 $A\otimes_K B$ 为弱拟三角的;当 A 为弱拟三角 Hopf 代数时,卷积代数 $\mathrm{Hom}_K(H,A)$ 是弱拟三角的.

关键词:Hopf 代数 张量代数 Yang-Baxte 方程

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Abstract:It is proved that if A,B are weakly-quasitriangular Hopf algebra,then $A\otimes_K B$ is weakly-quasitriangular, and if A is weakly-quasitriangular, then $\mathrm{Hom}_K(H,A)$ is also weakly-quasitriangular.

Key words:Hopf algebra,tensor algebra,Yang-Baxte equation

量子 Yang-Baxter 方程是杨振宁[1]和 Baxter[2]在物理领域中提出的新概念. 在数学领域中,一个弱拟三角 Hopf 代数能解一个量子 Yang-Baxter 方程[3]. 本文在文献[3]的基础上讨论张量代数 $A\otimes_K B$ 和卷积代数 $\mathrm{Hom}_K(H,A)$ 的弱拟三角结构. 文中的 Hopf 代数固定在基本域 K 上,记号 $\otimes_K$ 简写为 $\otimes$, $\mathrm{Hom}_K(H,A)$ 简写为 $\mathrm{Hom}(H,A)$,其它符号参考文献[4].

1 基本定义

定义 1[3] 一个弱拟三角 Hopf 代数(或双代数)是一对 Hopf 代数(或双代数)(H,R),这里 H 为 Hopf 代数(双代数),$R=\sum R_1\otimes R_2\in H\otimes H$,且满足:

(GQT.1)$R\Delta(h)=T\Delta(h)R,\forall h\in H$,

(GQT.2)$R_{(12)}R_{(13)}R_{(23)}=R_{(23)}R_{(13)}R_{(12)}$,

其中 T 是一个扭转变换,$T(a\otimes b)=b\otimes a,\forall a,b\in H$,$R_{(12)}=\sum R_1\otimes R_2\otimes 1$,$R_{(13)}=\sum R_1\otimes 1\otimes R_2$,$R_{(23)}=\sum 1\otimes R_1\otimes R_2$.

定义 2[5] 设 H,A 为有限维的 Hopf 代数,则 $(\mathrm{Hom}_K(H,A),*,\mu,\Delta_*,\varepsilon,s_*)$ 构成 K 上的 Hopf 代数,称为卷积 Hopf 代数. 其中 $*,\mu,\Delta_*,\varepsilon,s_*$ 分别定义为 $*:\mathrm{Hom}(H,A)\otimes\mathrm{Hom}(H,A)\longrightarrow\mathrm{Hom}(H,A)$,$*(f\otimes g)=M_A(f\otimes g)\Delta_H$;$\mu:K\longrightarrow\mathrm{Hom}(H,A)$,$\mu(1_K)=\mu_A\varepsilon_H$;$\Delta_*:\mathrm{Hom}(H,A)\longrightarrow\mathrm{Hom}(H,A)\otimes\mathrm{Hom}(H,A)$,$\Delta_*(f)=\Delta_A fM_H$;$\varepsilon:\mathrm{Hom}(H,A)\longrightarrow K$,$\varepsilon(f)=\varepsilon_A f\mu_H(1_K)$;$s_*:\mathrm{Hom}(H,A)\longrightarrow\mathrm{Hom}(H,A)$,$s_*(f)=s_A f s_H$.

2 主要结论

若(A,R^A),(B,R^B)均为弱拟三角 Hopf 代数,定义 $R^{A\otimes B}=\sum R^A_{(1)}\otimes R^B_{(1)}\otimes R^A_{(2)}\otimes R^B_{(2)}=(I\otimes T\otimes I)(R^A\otimes R^B)$,那么,$R^{A\otimes B}\in A\otimes B\otimes A\otimes B$.

定理 1 若(A,R^A),(B,R^B)均为弱拟三角 Hopf 代数,那么$(A\otimes B,R^{A\otimes B})$为弱拟三角 Hopf 代数.

证明 对 $\forall a\otimes b\in A\otimes B$,

$R^{A\otimes B}\Delta_{A\otimes B}(a\otimes b)=[(I\otimes T\otimes I)(R^A\otimes R^B)][(I\otimes T\otimes I)(\Delta_A\otimes\Delta_B)(a\otimes b)]=(I\otimes T\otimes I)[(R^A\otimes R^B)(\Delta_A\otimes\Delta_B)(a\otimes b)]=(I\otimes T\otimes I)(R^A\Delta_A(a)\otimes R^B\Delta_B(b))=(I\otimes T\otimes I)(T\Delta_A(a)\cdot R^A\otimes T\Delta_B(b)R^B)=(I\otimes T\otimes I)[(T\Delta_A(a)\otimes T\Delta_B(b))(R^A\otimes R^B)]=(I\otimes T\otimes I)(T\Delta_A(a)\otimes T\Delta_B(b))R^{A\otimes B}=(I\otimes T\otimes I)(T\Delta_A\otimes T\Delta_B)(a\otimes$

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$b)R^{A\otimes B} = T\Delta_{A\otimes B}(a \otimes b)R^{A\otimes B}$,

$R^{A\otimes B}_{(12)}R^{A\otimes B}_{(13)}R^{A\otimes B}_{(23)} = (I \otimes T \otimes T \otimes I)(I \otimes I \otimes T \otimes I \otimes I)[(R^A_{(12)} \otimes R^B_{(12)})](I \otimes T \otimes T \otimes I)(I \otimes I \otimes T \otimes I \otimes I)[(R^A_{(13)} \otimes R^B_{(13)})](I \otimes T \otimes T \otimes I) \cdot (I \otimes I \otimes T \otimes I \otimes I)[(R^A_{(23)} \otimes R^B_{(23)})] = (I \otimes T \otimes T \otimes I)(I \otimes I \otimes T \otimes I \otimes I)[(R^A_{(12)} \otimes R^B_{(12)})(R^A_{(13)} \otimes R^B_{(13)})(R^A_{(23)} \otimes R^B_{(23)})] = (I \otimes T \otimes T \otimes I)(I \otimes I \otimes T \otimes I \otimes I)(R^A_{(12)}R^A_{(13)}R^A_{(23)} \otimes R^B_{(12)}R^B_{(13)}R^B_{(23)}) = (I \otimes T \otimes T \otimes I)(I \otimes I \otimes T \otimes I \otimes I)(R^A_{(23)}R^A_{(13)}R^A_{(12)} \otimes R^B_{(23)}R^B_{(13)}R^B_{(12)}) = (I \otimes T \otimes T \otimes I)(I \otimes I \otimes T \otimes I \otimes I)[(R^A_{(23)} \otimes R^B_{(23)})(R^A_{(13)} \otimes R^B_{(13)})(R^A_{(12)} \otimes R^B_{(12)})] = (I \otimes T \otimes T \otimes I)(I \otimes I \otimes T \otimes I \otimes I)(R^A_{(23)} \otimes R^B_{(23)})(I \otimes T \otimes T \otimes I)(I \otimes I \otimes T \otimes I \otimes I)(R^A_{(13)} \otimes R^B_{(13)})(I \otimes T \otimes T \otimes I)(I \otimes I \otimes T \otimes I \otimes I)(R^A_{(12)} \otimes R^B_{(12)}) = R^{A\otimes B}_{(23)}R^{A\otimes B}_{(13)}R^{A\otimes B}_{(12)}$,

所以$(A \otimes B, R^{A\otimes B})$为弱拟三角 Hopf 代数.

设H,A为有限维的Hopf代数,且(A,R)为一个弱拟三角Hopf代数,其中$R = \sum R_{(1)} \otimes R_{(2)} \in A \otimes A$. 再令$R^* = \sum R^*_{(1)} \otimes R^*_{(2)} \in \mathrm{Hom}(H,A) \otimes \mathrm{Hom}(H,A)$, 对$\forall x,y \in H$,$R^*(x \otimes y) = \sum R^*_{(1)}(x) \otimes R^*_{(2)}(y) = \sum \varepsilon_H(x)R_{(1)} \otimes \varepsilon_H(y)R_{(2)}$.

定理2 若H,A为有限维的Hopf代数,其中H是交换的,且(A,R)为一个弱拟三角Hopf代数,则$(\mathrm{Hom}(H,A),R^*)$也是弱拟三角Hopf代数.

证明 对$\forall a,b \in H$,$\forall f \in \mathrm{Hom}(H,A)$,

$(R^* * \Delta_*(f))(a \otimes b) = M_{A\otimes A}(R^* \otimes \Delta_*(f)) \cdot \Delta_{H\otimes H}(a \otimes b) = \sum M_{A\otimes A}(R^* \otimes \Delta_*(f))(a_1 \otimes b_1 \otimes a_2 \otimes b_2) = \sum M_{A\otimes A}(R^*(a_1 \otimes b_1) \otimes \Delta_*(f)(a_2 \otimes b_2)) = \sum R^*(a_1 \otimes b_1)\Delta_*(f)(a_2 \otimes b_2) = \sum (\varepsilon_H(a_1)R_{(1)} \otimes \varepsilon_H(b_1)R_{(2)})((f(a_2b_2))_1 \otimes (f(a_2b_2))_2) = \sum \varepsilon_H(a_1)R_{(1)}(f(a_2b_2))_1 \otimes \varepsilon_H(b_1)R_{(2)}(f(a_2b_2))_2 = \sum R_{(1)}(f(ab))_1 \otimes R_{(2)}(f(ab))_2 = \sum (f(ab))_2R_{(1)} \otimes (f(ab))_1R_{(2)} = \sum f(ba)_2R_{(1)} \otimes f(ba)_1R_{(2)} = \sum ((f(b_1a_1))_2 \otimes (f(b_1a_1))_1)(\varepsilon_H(a_2)R_{(1)} \otimes \varepsilon_H(b_2)R_{(2)}) = \sum T\Delta_*(f)(a_1 \otimes b_1)R_*(a_2 \otimes b_2) = (T\Delta_*(f) * R_*)(a \otimes b)$,

所以$R^* * \Delta_*(f) = T\Delta_*(f) * R_*$.

$\forall x,y,z \in H$,

$R^*_{(12)}R^*_{(13)}R^*_{(23)}(x \otimes y \otimes z) = \sum (R^*_{(1)} * r^*_{(1)}) \otimes (R^*_{(2)} * l^*_{(1)}) \otimes (r^*_{(2)} * l^*_{(2)})(x \otimes y \otimes z) = \sum (R^*_{(1)} * r^*_{(1)})(x) \otimes (R^*_{(2)} * l^*_{(1)}(y) \otimes (r^*_{(2)} * l^*_{(2)})(z) = \sum [M_{A\otimes A}(R^*_{(1)} \otimes r^*_{(1)})\Delta_H(x)] \otimes [M_{A\otimes A}(R^*_{(2)} \otimes l^*_{(1)})\Delta_H(y) \otimes [M_{A\otimes A}(r^*_{(2)} \otimes l^*_{(2)})\Delta_H(z)] = \sum [M_{A\otimes A}(R^*_{(1)} \otimes r^*_{(1)})(x_1 \otimes x_2) \otimes [M_{A\otimes A}(R^*_{(2)} \otimes l^*_{(1)})(y_1 \otimes y_2)] \otimes [M_{A\otimes A}(r^*_{(2)} \otimes l^*_{(2)})(z_1 \otimes z_2)] = \sum \varepsilon_H(x_1)R_{(1)}\varepsilon_H(x_2)r_{(1)} \otimes \varepsilon_H(y_{(1)})R_{(2)}\varepsilon_H(y_2)l_{(1)} \otimes \varepsilon_H(z_1)r_{(2)}\varepsilon_H(z_2)l_{(2)} = \sum \varepsilon_H(x)R_{(1)}r_{(1)} \otimes \varepsilon_H(y)R_{(2)} \cdot l_{(1)} \otimes \varepsilon_H(z)r_{(2)}l_{(2)} = \sum \varepsilon_H(x)\varepsilon_H(y)\varepsilon_H(z) \cdot R_{(1)}r_{(1)} \otimes R_{(2)}l_{(1)} \otimes r_{(2)}l_{(2)} = \sum \varepsilon_H(x)\varepsilon_H(y) \cdot \varepsilon_H(z)r_{(1)}R_{(1)} \otimes l_{(1)}R_{(2)} \otimes l_{(2)}r_{(2)} = \sum \varepsilon_H(x)r_{(1)}R_{(1)} \otimes \varepsilon_H(y)l_{(1)}R_{(2)} \otimes \varepsilon_H(z)l_{(2)}r_{(2)} = \sum (r^*_{(1)} * R^*_{(1)}) \otimes (l^*_{(1)} * R^*_{(2)}) \otimes (l^*_{(2)} * r^*_{(2)})(x \otimes y \otimes z) = R^*_{(23)}R^*_{(13)}R^*_{(12)}(x \otimes y \otimes z)$,

所以$R^*_{(12)}R^*_{(13)}R^*_{(23)} = R^*_{(23)}R^*_{(13)}R^*_{(12)}$. 定理2成立.

若$(\mathrm{Hom}(H,A),R^*)$为一个弱拟三角Hopf代数,$R^* = \sum R^*_{(1)} \otimes R^*_{(2)} \in \mathrm{Hom}(H,A) \otimes \mathrm{Hom}(H,A)$. 定义$R = R^*(1_H \otimes 1_H) = \sum R^*_{(1)}(1_H) \otimes R^*_{(2)}(1_H) \in A \otimes A$.

定理3 若H,A为有限维的Hopf代数,且$(\mathrm{Hom}(H,A),R^*)$为一个弱拟三角Hopf代数,则(A,R)也是弱拟三角Hopf代数.

证明 对$\forall a \in A$,显然存在一个$f \in \mathrm{Hom}(H,A)$,使得$f(1_H) = a$,那么有

$R\Delta(a) = R^*(1_H \otimes 1_H)\Delta_A(f(1_H)) = (R^*(1_H \otimes 1_H))\Delta_*(f)(1_H \otimes 1_H)) = M_{A\otimes A}(R \otimes \Delta_*(f))(1_H \otimes 1_H \otimes 1_H \otimes 1_H) = M_{A\otimes A}(R \otimes \Delta_*(f))\Delta_{H\otimes H}(1_H \otimes 1_H) = R^* * \Delta_*(f)(1_H \otimes 1_H)) = (T\Delta_*(f) * R^*) \cdot (1_H \otimes 1_H) = (T\Delta_*(f)(1_H \otimes 1_H))(R^*(1_H \otimes 1_H)) = \sum ((f(1_H))_2) \otimes ((f(1_H))_1)R = T\Delta(f(1_H))R = T\Delta(a)R$.

$R_{(12)} = \sum R_{(1)} \otimes R_{(2)} \otimes 1_A = \sum R^*_{(1)}(1_H) \otimes R^*_{(2)}(1_H) \otimes 1_A$,

$R_{(13)} = \sum r^*_{(1)}(1_H) \otimes 1_A \otimes r^*_{(2)}(1_H)$,

$R_{(23)} = \sum 1_A \otimes l^*_{(1)}(1_H) \otimes l^*_{(2)}(1_H)$,

$R_{(12)}R_{(13)}R_{(23)} = \sum (R^*_{(1)}(1_H) \otimes R^*_{(2)}(1_H) \otimes 1_A)(r^*_{(1)}(1_H) \otimes 1_A \otimes r^*_{(2)}(1_H))(1_A \otimes l^*_{(1)}(1_H) \otimes l^*_{(2)}(1_H)) = \sum (R^*_{(1)}(1_H)r^*_{(1)}(1_H)) \otimes (R^*_{(2)}(1_H) \cdot l^*_{(1)}(1_H)) \otimes r^*_{(2)}(1_H)l^*_{(2)}(1_H) = \sum (R^*_{(1)} * r^*_{(1)})(1_H) \otimes (R^*_{(2)} * l^*_{(1)})(1_H) \otimes (r^*_{(2)} * l^*_{(2)})(1_H) = \sum ((R^*_{(1)} * r^*_{(1)}) \otimes (R^*_{(2)} * l^*_{(1)}) \otimes (r^*_{(2)} * l^*_{(2)}))(1_H \otimes 1_H \otimes 1_H) = \sum ((r^*_{(1)} * R^*_{(1)}) \otimes (l^*_{(1)} * R^*_{(2)}) \otimes (l^*_{(2)} * r^*_{(2)}))(1_H \otimes 1_H \otimes 1_H) = \sum ((r^*_{(1)}(1_H)R^*_{(1)}(1_H)) \otimes (l^*_{(1)}(1_H) \cdot$

$R^{*}_{(2)}(1_H))\otimes(l^{*}_{(2)}(1_H)r^{*}_{(2)}(1_H)))=\sum(1_A\otimes l^{*}_{(1)}\cdot(1_H)\otimes l^{*}_{(2)}(1_H))(r^{*}_{(1)}(1_H)\otimes 1_A\otimes r^{*}_{(2)}(1_H))(R^{*}_{(1)}\cdot(1_H)\otimes R^{*}_{(2)}(1_H)\otimes 1_A)=R_{(23)}R_{(13)}R_{(12)}$,

所以,定理 3 成立.

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