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On the Boundedness Nature of Positive Solutions of the Difference Equation $x_n=\frac{A+x_{n-1}^p}{B+x_{n-k}^p}$ *

关于差分方程 $x_n=\frac{A+x_{n-1}^p}{B+x_{n-k}^p}$ 正解的有界性

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Abstract The difference equation $x_n=\frac{A+x_{n-1}^p}{B+x_{n-k}^p}$, $n=0,1,2,\cdots$, are considered in this paper, where $k\geqslant 2$ and $A,B,p\in(0,+\infty)$. We show that if $p^{k-1}\geqslant\frac{k^k}{(k-1)^{k-1}}$, then this equation has positive unbounded solutions, and if $p^{k-1}<\frac{k^k}{(k-1)^{k-1}}$, then every positive solution of this equation is bounded.

Key words difference equation, boundedness, positive solution

摘要: 证明差分方程 $x_n=\frac{A+x_{n-1}^p}{B+x_{n-k}^p}$, $n=0,1,2,\cdots$, (其中 $k\geqslant 2$, $A,B,p\in(0,+\infty)$)在 $p^{k-1}\geqslant\frac{k^k}{(k-1)^{k-1}}$ 时,有无界的解,并且当 $p^{k-1}<\frac{k^k}{(k-1)^{k-1}}$ 时,每个正解都有界.

关键词: 差分方程 有界性 正解
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Recently there have been published quite a lot of works concerning the behavior of positive solutions of systems of rational difference equations[1～4]. These results are not only valuable in their own right, but can also provide insight into their differential counterparts.

In reference [5], Stević studied difference equation

$$x_{n+1}=\max\{A,\frac{x_n^p}{x_{n-1}^p}\},n=0,1,2,\cdots, \quad (1)$$

where $A,p\in(0,+\infty)$ and the initial values x_{-1}, $x_0\in(0,+\infty)$. He showed that (i) If $p\geqslant 4$, then equation (1) has positive unbounded solutions, and (ii) if $p<4$, then every positive solution of equation (1) is bounded.

In reference [6], Stević investigated the difference equation

$$x_n=A+\frac{x_{n-1}^p}{x_{n-k}^p},n=0,1,2,\cdots, \quad (2)$$

where $k\in\{2,\cdots\}$, $A,p\in(0,+\infty)$ and the initial values x_{-k}, $x_{-k+1},\cdots,x_{-1}\in(0,+\infty)$. He showed that (i) If $p^{k-1}\geqslant\frac{k^k}{(k-1)^{k-1}}$, then equation (2) has positive unbounded solutions, and (ii) if $p^{k-1}<\frac{k^k}{(k-1)^{k-1}}$, then every positive solution of equation (2) is bounded.

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In refrence [7], Kulenovic et al. showed that every positive solution of the difference equation

$$x_n = \frac{A + x_{n-1}^p}{B + x_{n-k}^p}, n = 0, 1, \cdots \quad (3)$$

is bounded when $p = 1$, where $k \geqslant 2$ and $A, B, p \in (0, +\infty)$. In this paper, we also conside the equation (3).

1 Lemma

Lemma 1 Consider the difference equation

$$x_n \geqslant \max\{A, \frac{x_{n-1}^p}{x_{n-k}^p}\}, n = 0, 1, \cdots. \quad (4)$$

Assume that $k \geqslant 2$ and $A, p \in (0, +\infty)$ with $p^{k-1} \geqslant \frac{k^k}{(k-1)^{k-1}}$. Then there exists $x_{-k}, x_{-k+1}, \cdots, x_{-1}$ large enough such that every positive solution of equation (4) with the initial values $x_{-k}, x_{-k+1}, \cdots, x_{-1}$ is unbounded.

The proof of lemma 1 is quite similar to one of theorem 1 in reference [6], and hence is omitted.

2 Main result

Theorem 1 (i) If $p^{k-1} \geqslant \frac{k^k}{(k-1)^{k-1}}$, then equation (3) has positive unbounded solutions.

(ii) If $p^{k-1} < \frac{k^k}{(k-1)^{k-1}}$, then every positive solution of equation (3) is bounded.

Proof (i) Let $\{x_n\}_{n=-k}^{\infty}$ is a positive solution of equation (3). There are two cases to be considered.

Case 1 $A \geqslant B$. Then we have

$$x_n = \frac{A + x_{n-1}^p}{B + x_{n-k}^p} \geqslant \frac{B + x_{n-1}^p}{B + x_{n-k}^p}. \quad (5)$$

It follows from equation (5) that

$$x_n^p + B \geqslant (\frac{B + x_{n-1}^p}{B + x_{n-k}^p})^p.$$

Set $y_n = x_n^p + B$, then we have

$$y_n \geqslant \max\{B, \frac{y_{n-1}^p}{y_{n-k}^p}\}. \quad (6)$$

It follows from lemma 1 that there exists $y_{-k}, y_{-k+1}, \cdots, y_{-1} \in (B, +\infty)$ such that every positive solution of equation (6) with the initial values $y_{-k}, y_{-k+1}, \cdots, y_{-1}$ is unbounded, which implies that the positive solution of equation (3) with the initial values $x_i = (y_{-i} - B)^{\frac{1}{p}} (1 \leqslant i \leqslant k)$ is unbounded.

Case 2 $A < B$. Then we have

$$\frac{B}{A} x_n = \frac{\frac{B}{A}(A + x_{n-1}^p)}{B + x_{n-k}^p} \geqslant \frac{B + x_{n-1}^p}{B + x_{n-k}^p}. \quad (7)$$

It follows from equaton (7) that

$$(\frac{B}{A})^p (x_n^p + B) \geqslant (\frac{B + x_{n-1}^p}{B + x_{n-k}^p})^p. \quad (8)$$

Set $y_n = (\frac{B}{A})^p (x_n^p + B)$, we obtain that

$$y_n \geqslant \max\{B, \frac{y_{n-1}^p}{y_{n-k}^p}\}. \quad (9)$$

It follows from lemma 1 that there exists $y_{-k}, y_{-k+1}, \cdots, y_{-1} \in (B^{p+1}/A^p, +\infty)$ such that every positive solution of equation (9) with the initial values $y_{-k}, y_{-k+1}, \cdots, y_{-1}$ is unbounded, which also implies that equation (3) has positive unbounded solutions. From all above mentioned the theorem 1 (i) is proven.

(ii) Let $\{x_n\}_{n=-k}^{\infty}$ is a positive solution of equation (3). There are also two cases to be considered.

Case 1 $A \leqslant B$. It follows from equation (3) that

$$x_n \leqslant \frac{B + x_{n-1}^p}{B + x_{n-k}^p}. \quad (10)$$

By repeating use of the recurrence relation in equation (10), we obtain that

$$x_n \leqslant \frac{B}{B + x_{n-k}^p} + \frac{x_{n-1}^p}{B + x_{n-k}^p} \leqslant \frac{B}{B + x_{n-k}^p} + \frac{(B + x_{n-2}^p)^p}{(B + x_{n-k}^p)(B + x_{n-k-1}^p)^p} \leqslant \frac{B}{B + x_{n-k}^p} + (\frac{B}{(B + x_{n-k}^p)^{\frac{1}{p}}(B + x_{n-k-1}^p)} + \frac{(B + x_{n-3}^p)^p}{(B + x_{n-k}^p)^{\frac{1}{p}}(B + x_{n-k-1}^p)(B + x_{n-k-2}^p)^p})^p \leqslant$$

$$\frac{B}{B + x_{n-k}^p} + (\frac{B}{(B + x_{n-k}^p)^{\frac{1}{p}}(B + x_{n-k-1}^p)} + \cdots + (\frac{B}{\prod_{j=0}^{k-2}(B + x_{n-k-j}^p)^{1/p^{k-2-j}}} + \frac{(B + x_{n-k}^p)^{p - \frac{1}{p^{k-2}}}}{\prod_{j=1}^{k-1}(B + x_{n-k-j}^p)^{1/p^{k-2-j}}})^p \cdots)^p = \frac{B}{B + x_{n-k}^p} +$$

$$(\frac{B}{(B + x_{n-k}^p)^{\frac{1}{p}}(B + x_{n-k-1}^p)} + \cdots + \frac{B}{\prod_{j=0}^{k-2}(B + x_{n-k-j}^p)^{a_0^{(j)}}} + \frac{(B + x_{n-k}^p)^{p - a_0^{(0)}}}{(\prod_{j=1}^{k-2}(B + x_{n-k-j}^p)^{a_0^{(j)}})(B + x_{n-2k+1}^p)^p})^p \cdots)^p =$$

$$\frac{B}{B + x_{n-k}^p} + (\frac{B}{(B + x_{n-k}^p)^{\frac{1}{p}}(B + x_{n-k-1}^p)} + \cdots + (\frac{B}{\prod_{j=0}^{k-2}(B + x_{n-k-j}^p)^{a_0^{(j)}}} +$$

$$
\Big(\frac{B+x_{n-k}^{p}}{\big(\prod_{j=1}^{k-2}(B+x_{n-k-j}^{p})^{a_0^{(j)}/(p-a_0^{(0)})}\big)(B+x_{n-2k+1}^{p})^{p/(p-a_0^{(0)})}}\Big)^{p-a_0^{(0)}}\Big)^{p}\cdots\Big)^{p}=\frac{B}{B+x_{n-k}^{p}}+
$$

$$
\Big(\frac{B}{(B+x_{n-k}^{p})^{\frac{1}{p}}(B+x_{n-k-1}^{p})}+\cdots+
$$

$$
\Big(\frac{B}{\prod_{j=0}^{k-2}(B+x_{n-k-j}^{p})^{a_0^{(j)}}}+
$$

$$
\Big(\frac{B+x_{n-k}^{p}}{\prod_{j=0}^{k-2}(B+x_{n-k-j-1}^{p})^{a_1^{(j)}}}\Big)^{p-a_0^{(0)}}\Big)^{p}\cdots\Big)^{p}\leqslant
$$

$$
\frac{B}{B+x_{n-k}^{p}}+\Big(\frac{B}{(B+x_{n-k}^{p})^{\frac{1}{p}}(B+x_{n-k-1}^{p})}\Big)+\cdots+
$$

$$
\Big(\frac{B}{\prod_{j=0}^{k-2}(B+x_{n-k-m-j}^{p})^{a_m^{(j)}}}+
$$

$$
\frac{(B+x_{n-k-m}^{p})^{p-a_m^{(0)}}}{\big(\prod_{j=1}^{k-2}(B+x_{n-k-m-j}^{p})^{a_m^{(j)}}\big)(B+x_{n-2k+1-m}^{p})^{p}}
$$

$$
\Big)^{p-a_{m-1}^{(0)}}\cdots\Big)^{p}. \qquad (11)
$$

For each $k\geqslant 2$ and every $n\geqslant 2k+m-1$, where the sequences $a_m^{(j)}$, $j\in\{0,1,\cdots,k-2\}$ are defined by

$$
a_{m+1}^{(0)}=\frac{a_m^{(1)}}{p-a_m^{(0)}},\cdots,a_{m+1}^{(k-3)}=\frac{a_m^{(k-2)}}{p-a_m^{(0)}},a_{m+1}^{(k-2)}=\frac{p}{p-a_m^{(0)}}, \qquad (12)
$$

with

$$
a_0^{(j)}=\frac{1}{p^{k-2-j}},j\in\{0,1,\cdots,k-2\}.
$$

Since

$$
a_0^{(j)}=\frac{1}{p^{k-2-j}}<\frac{1}{p^{k-2-j}-p^{-j}}=a_1^{(j)},
$$

and by using equation (12), it is easy to see that the sequences $a_m^{(j)}$, $j\in\{0,1,\cdots,k-2\}$ are strictly increasing. From equation (8), we also have that

$$
a_{m+1}^{(0)}=\frac{p}{(p-a_m^{(0)})(p-a_{m-1}^{(0)})\cdots(p-a_{m-k+2}^{(0)})}. \qquad (13)
$$

Hence if $a_m^{(0)}<p$ for every $m\in$ N, then there is a finite limit $\lim_{m\to\infty}a_m^{(0)}=x^{*}\in(0,p]$, and it follows from equation (13) that x^{*} is a solution of the equation

$$
f(x)=x(p-x)^{k-1}-p=0.
$$

Since $f'(x)=(p-x)^{k-2}(p-kx)$, we see that the function attains its maximum on the interval $[0,p]$, at $x=p/k$. Since

$$
f(p/k)=\frac{(k-1)^{k-1}}{k^{k}}p^{k}-p=p\frac{(k-1)^{k-1}}{k^{k}}\cdot\Big(p^{k-1}-\frac{k^{k}}{(k-1)^{k-1}}\Big),
$$

we see that if $p^{k-1}<\frac{k^{k}}{(k-1)^{k-1}}$, then the function $f(x)$ has no solutions on the interval $(0,p]$, which is a contradiction. Hence, this is the first $k_0\in$ N such that $a_{k_0-1}^{(0)}<p$ and $a_{k_0}^{(0)}\geqslant p$. From this and equation (11) with $k=k_0$, it follows that

$$
x_n\leqslant 1+\Big(\frac{1}{B^{1/p}}+\cdots+\Big(\frac{1}{B^{\sum_{j=0}^{k-2}a_{k_0-1}^{(j)}}}+\frac{1}{B^{\sum_{j=0}^{k-2}a_{k_0}^{(j)}}}\Big)^{p-a_{k_0-1}^{(0)}}\cdots\Big)^{p},
$$

for $n\geqslant 2k+k_0-1$. Then all positive solutions of equation (3) are bounded.

Case 2 $A>B$. It follows from equation (3) that

$$
x_n\leqslant\frac{\frac{A}{B}(B+x_{n-1}^{p})}{B+x_{n-k}^{p}}.
$$

The following proof is the same as in the case 1, hence it will be omitted. From all mentioned above the theorem 1 is complete.

References

[1] Berenhaut K, Foley J, Stević S. Quantitative bound for the recursive sequence $y_{n+1}=A+\frac{y_n}{y_{n-k}}$ [J]. Appl Math Lett, 2006, 19(9): 983–989.

[2] Berenhaut K, Stević S. The behavior of the positive solutions of the difference equation $x_n=A+(\frac{x_{n-2}}{x_{n-1}})^{p}$ [J]. J Differ Equations Appl, 2006, 12(9): 909–918.

[3] Kent C M, Radin M A. On the boundedness nature of positive solutions of the difference equation $x_{n+1}=\max\left\{\frac{A_n}{x_n},\frac{B_n}{x_{n-1}}\right\}$, with periodic parameters [J]. Dyn Contin Discrete Impuls Syst Ser B Appl Algorithms, 2003, 11–15.

[4] Stević S. On the recursive sequence $x_{n+1}=1+x_n^{p}/x_{n-1}^{p}$ [J]. Discrete Dyn Nat Soc, 2007, 9.

[5] Stević S. On the recursive sequence $x_{n+1}=\max\{c,x_n^{p}/x_{n-1}^{p}\}$ [J]. Appl Math Lett, 2008, 21(8): 791–796.

[6] Stević S. The boundedness character of positive solutions of the difference equation $x_n=A+\frac{x_{n-1}^{p}}{x_{n-k}^{p}}$ [J]. Nonlinear Analysis, In press.

[7] Kulenovic M R S, Ladas G. Dynamics of second order rational difference equations, with open problems and conjectures [J]. Chapman and Hall/CRC, 2002.

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