DOI: 10.13656/j.cnki.gxkx.2008.03.003
广西科学 Guangxi Sciences 2008, 15(3): 211-215

On the Upper Bounds for Vertex Folkman Numbers*
顶点 Folkman数的上界

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Abstract Some new inequalities on the upper bounds for vertex Folkman numbers are proven in this paper. In particular, we prove the following result by constructive method: for any real number r that satisfies $0<r<\frac{1}{2}\log_2 3-\frac{3}{4}$, there are $N(r)>0$ and $c(r)>0$ such that $F_v(k,k;k+1)\leqslant c(r)(k-1)^{\frac{1}{4}\log_2(k-1)-r}$ for any $k\geqslant N(r)$, in which both $N(r)$ and $c(r)$ are constants only depending on r.
Key words vertex Folkman number, upper bound, composition
摘要:证明关于顶点 Folkman数上界的新不等式.特别地,用构造性方法证明:对于任意满足 $0<r<\frac{1}{2}\log_2 3-\frac{3}{4}$的实数 r,存在 $N(r)>0$和 $c(r)>0$使得 $F_v(k,k;k+1)\leqslant c(r)(k-1)^{\frac{1}{4}\log_2(k-1)-r}$对任意的 $k\geqslant N(r)$成立,其中 $N(r)$和 $c(r)$都是只依赖于 r的常数.
关键词:顶点 Folkman数,上界,合成图
中图法分类号: O157.5 文献标识码: A 文章编号: 1005-9164(2008)03-0211-05

For an undirected, simple graph G, and positive integers $a_1,\cdots,a_k$, we write $G\rightarrow(a_1,\cdots,a_k)^v$ ($G\rightarrow(a_1,\cdots,a_k)^e$) if and only if for every vertex (edge) k-coloring of G, there exists a monochromatic K_{a_i}, for some color $i\in\{1,\cdots,k\}$.

For positive integers $a_1,\cdots,a_k$ and $p>\max\{a_1,\cdots,a_k\}$, let

$$\boldsymbol{F}_v(a_1,\cdots,a_k;p)=\{G: G\rightarrow(a_1,\cdots,a_k)^v, K_p\not\subseteq G\},$$
$$\boldsymbol{F}_e(a_1,\cdots,a_k;p)=\{G: G\rightarrow(a_1,\cdots,a_k)^e, K_p\not\subseteq G\}.$$

The graphs in $\boldsymbol{F}_v(a_1,\cdots,a_k;p)$ are called vertex Folkman graphs, and the graphs in $\boldsymbol{F}_e(a_1,\cdots,a_k;p)$ are called edge Folkman graphs.

In 1970, Folkman[1] showed that for all r,l and $p>\max\{r,l\}$ the families $\boldsymbol{F}_v(r,l;p)$ and $\boldsymbol{F}_e(r,l;p)$ are nonempty. Folkman's method worked only for two colors. Folkman's theorem was generalized to multicolor case in reference[2,3].

Someone could ask what is the minimum number of vertices in a vertex or edge Folkman graph, which leads to the notion of Folkman numbers.

For positive integers $a_1,\cdots,a_k$ and $p>\max\{a_1,\cdots,a_k\}$, the vertex (edge) Folkman number is defined as

$$F_v(a_1,\cdots,a_k;p)=\min\{|V(G)|: G\rightarrow(a_1,\cdots,a_k)^v, K_p\not\subseteq G\},$$
$$F_e(a_1,\cdots,a_k;p)=\min\{|V(G)|: G\rightarrow(a_1,\cdots,a_k)^e, K_p\not\subseteq G\}.$$

Among all vertex Folkman numbers, $F_v(k,k;k+1)$ seems more interesting for many researchers. In this paper we will discuss the upper bound for it and get new upper bounds for vertex Folkman numbers based on composition of two graphs.

收稿日期: 2008-03-06
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* Partially Supported by the National Natural Science Fund of China (60563008), and the Basic Research Fund of Guangxi Academy of Sciences (080414).

1 Some known knowledges

Theorem 1[4] For k no less than 3, we have

$F_v(k,k;k+1) \leqslant \lfloor k!\, e \rfloor - 2$.

In reference[5], $F_v(k,k;k+1) \leqslant \lfloor 2k!\,(e-1) \rfloor - 2$ was proved. Theorem 1 in reference[4] was proved earlier than this inequality. reference [4] was in Russian and was not well known.

Theorem 2[6] For all integer p no less than 2, we have

$F_v(p+1,p+1;p+2) \leqslant (p+1)F_v(p,p;p+1)$.

Corollary 1[6] For integer p no less than 4, we have $F_v(p,p;p+1) \leqslant 1.46p!$.

We improved Theorem 2 as following.

Theorem 3[7] For integer k no less than 2, we have

$F_v(2k,2k;2k+1) \leqslant kF_v(2k-1,2k-1;2k) + 3k + 1$.

Theorem 4[7] Suppose k is an integer no less than 2, $H \in \boldsymbol{F}_v(2k,2k;2k+1)$, $\chi(H) = F_v(2k,2k;2k+1)$. Let $\{v_1\} \subset H$, $A \subset V(H) \setminus \{v_1\}$, G_1 be the subgraph of H induced by A and G_2 which is the subgraph of H induced by $V(H) \setminus (\{v_1\} \cup A)$. Suppose both G_1 and G_2 are K_{2k}-free. If x is the order of the maximum isomorphic induced subgraphs of G_1 and G_2, then we have

$F_v(2k+1,2k+1;2k+2) \leqslant (k+1)F_v(2k,2k;2k+1) - x + 3k + 2$.

2 Upper bounds for vertex Folkman numbers based on composition

We define the composition of simple graph of G, H and $G[H]$, as follow: its vertex set is $V(G) \times V(H)$; in $G[H]$, there is edge connecting different vertex (u,v) and (u',v'), if and only if $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$[8].

It is easy to know that $\chi(G[H]) = \chi(G) \circ \chi(H)$.

Theorem 5 Let $a_1,\cdots,a_k,b_1,\cdots,b_k,p,q$ be positive integers, $\max\{a_1,\cdots,a_k\} \leqslant p$, $\max\{b_1,\cdots,b_k\} \leqslant q$, then

$F_v(a_1b_1,\cdots,a_kb_k;pq+1) \leqslant F_v(a_1,\cdots,a_k;p+1) \circ F_v(b_1,\cdots,b_k;q+1)$.

Proof Suppose $G \in \boldsymbol{F}_v(a_1,\cdots,a_k;p+1)$, $\chi(G) = n_1 = F_v(a_1,\cdots,a_k;p+1)$, and $H \in \boldsymbol{F}_v(b_1,\cdots,b_k;q+1)$, $\chi(H) = n_2 = F_v(b_1,\cdots,b_k;q+1)$. Since G is K_{p+1}-free and H is K_{q+1}-free, we know $G[H]$ is K_{pq+1}-free. Now we will prove $G[H] \to (a_1b_1,\cdots,a_kb_k)^v$.

Suppose $V(G) = \{v_1,\cdots,v_{n_1}\}$ and $V(H) = \{v_1,\cdots,v_{n_2}\}$.

Let H' be isomorphic to H, $V(H') = \{v_1',\cdots,v_{n_2}'\}$, and for different $i,j \in \{1,\cdots,n_2\}$, $v_i'v_j' \in E(H')$, if and only if $v_iv_j \in V(H)$.

Now, we give $G[H]$ any vertex k-coloring, i.e., we color any vertex in $G[H]$ with a color among $\{\text{color } i \mid i \in \{1,\cdots\cdots,k\}\}$.

For any $j \in \{1,\cdots\cdots,n_2\}$, suppose $V_j = \{(u_i,v_j) \mid 1 \leqslant i \leqslant n_1\}$, So $V(G[H]) = \bigcup_{j=1}^{n_2} V_j$.

For any $j \in \{1,\cdots\cdots,n_2\}$, from the definition of $G[H]$, we know the subgraph of $G[H]$ induced by V_j is isomorphic to G. Therefore for any $j \in \{1,\cdots\cdots,n_2\}$, from $G \to (a_1,\cdots,a_k)^v$ we know there is monochromatic K_{a_x} in the subgraph of $G[H]$ induced by V_j for some color $x \in \{1,\cdots\cdots,k\}$. If there are more than one such x's, we can choose any one among them. Let $x = f(j)$. Now we give H' a vertex k-coloring, for any $j \in \{1,\cdots\cdots,n_2\}$, we color v_j' with color $f(j)$.

Since that H' is isomorphic to H, and $H \to (b_1,\cdots,b_k)^v$, there is monochromatic K_{b_y} in H' for some color $y \in \{1,\cdots\cdots k\}$.

From the program of the vertex coloring of graph H', we know this means that for the vertex coloring of $G[H]$, there are b_y ones among $\{1,\cdots\cdots,n_2\}$, say $\{z_1,\cdots\cdots,z_{b_y}\}$, such that for any $j \in \{z_1,\cdots\cdots,z_{b_y}\}$, there are a_y vertices in V_j that induce a complete subgraph in $G[H]$ of color y. So we find a complete subgraph of $G[H]$ in color y on a_yb_y vertices. Therefore we prove $G[H] \to (a_1b_1,\cdots a_kb_k)^v$.

We know the order of $G[H]$ is $F_v(a_1,\cdots,a_k;p+1) \circ F_v(b_1,\cdots,b_k;q+1)$, so we have

$F_v(a_1b_1,\cdots,a_kb_k;pq+1) \leqslant F_v(a_1,\cdots,a_k;p+1) \circ F_v(b_1,\cdots,b_k;q+1)$.

Corollary 2 Let a,b be positive integers no less than 2, $a \leqslant p$, $b \leqslant q$, then

$F_v(ab,ab;pq+1) \leqslant F_v(a,a;p+1)F_v(b,b;q+1)$.

In particular, from $F_v(2,2;3) = 5$ we have the following corollary that will be used later.

Corollary 3 Let a be a positive integer no less than 2, then

$F_v(2a,2a;2a+1) \leqslant 5F_v(a,a;a+1)$.

3 Constructive upper bound for vertex Folkman number $F_v(2k+1,2k+1;2k+2)$

Theorem 6 Suppose k is an integer no less than

$4, f(k) = \left\lceil \sqrt{\frac{k}{3}} \right\rceil$, then we have

$F_v(2k+1, 2k+1; 2k+2) \leqslant [2f(k)+1]F_v(k,k;k+1) + 3F_v(k+1, k+1; k+2) + k + 2f(k)$.

Proof Suppose $S \in \boldsymbol{F}_v(k,k;k+1)$, $g(S) = n = F_v(k,k;k+1)$, $V(S) = \{s_1, \cdots, s_n\}$, $T \in \boldsymbol{F}_v(k+1, k+1; k+2)$, $g(T) = F_v(k+1, k+1; k+2)$.

We will construct a graph G to prove the inequality in theorem 6.

Let $V(G) = V(G_1) \cup V(G_2) \cup V(S_0) \cup \bigcup_{i=1}^{3} V(T_i) \cup V(C)$.

From $g(S) = n = F_v(k,k;k+1)$, we know that there is $V(A) \subset V(S) \setminus \{s_1\}$ such that both the subgraph of S induced by $V(A)$ and the subgraph of S induced by $V(B) = V(S) \setminus (\{s_1\} \cup V(A))$ are K_k-free. We might suppose $V(A) = \{s_2, \cdots, s_{n_1}\}$ and $V(B) = \{s_{n_1+1}, \cdots, s_n\}$ as well.

Let $I = Z_{f(k)} = \{i \bmod f(k) \mid 1 \leqslant i \leqslant f(k)\}$.

For G_1, $V(G_1) = \bigcup_{i \in I} V(A_i) \cup \bigcup_{i \in I} V(B_i) \cup \{u_1(i) \mid i \in I\} \cup \{u_1'(i) \mid i \in I\}$, in which for any $i \in I$, $V(A_i) = \{u_2(i), \cdots, u_{n_1}(i)\}$, and $V(B_i) = \{u_{n_1+1}(i), \cdots, u_n(i)\}$.

$E(G_1) = \{(u_y(i), u_z(i)) \mid s_y s_z \in E(S), i \in I, 1 \leqslant y < z \leqslant n\} \cup \bigcup_{i=1}^{3} E_i(1)$, where

$E_1(1) = \{(u_1'(i), u_z(i)) \mid s_1 s_z \in E(S), i \in I, n_1 + 1 \leqslant z \leqslant n\}$,

$E_2(1) = \{(u_1'(i), u_z(i+1)) \mid s_1 s_z \in E(S), i \in I, 2 \leqslant z \leqslant n_1\}$,

$E_3(1) = \{(u_y(i), u_z(i+1)) \mid s_y s_z \in E(S), i \in I, n_1 + 1 \leqslant y \leqslant n, 2 \leqslant z \leqslant n_1\}$.

For any i satisfies $1 \leqslant i \leqslant f(k)$, let j be i mod $f(k)$, we set

$V(S_{2i-1}) = \{u_y(j) \mid 1 \leqslant y \leqslant n\}$,

$V(S_{2i}) = \{u_y(j) \mid n_1 + 1 \leqslant y \leqslant n\} \cup \{u_z(j+1) \mid 2 \leqslant z \leqslant n_1\} \cup \{u_1'(j)\}$.

We will construct G_2 similarly. Let $I' = \{i \bmod (2f(k)) \mid f(k) + 1 \leqslant i \leqslant 2f(k)\}$. Set $V(G_2) = \bigcup_{i \in I'} V(A_i) \cup \bigcup_{i \in I'} V(B_i) \cup \{u_1(i) \mid i \in I'\} \cup \{u_1'(i) \mid i \in I'\}$, in which $V(A_i) = \{u_2(i), \cdots, u_{n_1}(i)\}$ and $V(B_i) = \{u_{n_1+1}(i), \cdots, u_n(i)\}$ for any $i \in I'$.

Let G_2 be isomorphic to G_1, i. e., there is a bijection g such that for any i satisfies $1 \leqslant i \leqslant f(k)$, let j be i mod $f(k)$ and j' be $(i + f(k)) \bmod (2f(k))$,

$g(u_y(j)) = u_y(j')$, $1 \leqslant y \leqslant n$,

$g(u_1'(j)) = u_1'(j')$,

and $(g(u), g(v)) \in E(G_2)$ if and only if $(u,v) \in E(G_1)$.

For any i satisfies $2f(k) + 1 \leqslant i \leqslant 4f(k)$, let $V(S_i) = g(V(S_{i-2f(k)}))$.

We define U to be the subgraph of G induced by $V(U)$ for any $V(U) \subset V(G)$.

For any i satisfies $1 \leqslant i \leqslant 4f(k)$, we have that S_i is isomorphic to S. Let S_0 be isomorphic to S.

For any i satisfies $1 \leqslant i \leqslant 3$, let the subgraph of graph G induced by $V(T_i)$ be isomorphic to T.

Let the subgraph of graph G induced by $V(C)$ be isomorphic to K_k, the complete graph of order k.

Let $E_1 = E(G_1) \cup E(G_2) \cup E(S_0) \cup \bigcup_{i=1}^{3} E(T_i) \cup E(C)$,

$E_2 = E(T_3, C) \cup E(\bigcup_{i=1}^{3} T_i, S_0) \cup \bigcup_{i=1}^{2} E(G_i, T_i)$.

In any part in $E_2 = E(T_3, C) \cup E(\bigcup_{i=1}^{3} T_i, S_0) \cup \bigcup_{i=1}^{2} E(G_i, T_i)$, $E(U, U') = \{(u, u') \mid u \in U, u' \in U'\}$.

Suppose $1 \leqslant i \leqslant 2f(k)$, $2f(k) + 1 \leqslant j \leqslant 4f(k)$, and at least one of them is odd integer, then we have at least $3[f(k)]^2$ choices to take (S_i, S_j), in which $S_i \subset G_1$, $S_j \subset G_2$.

From $f(k) = \left\lceil \sqrt{\frac{k}{3}} \right\rceil$ we have $3[f(k)]^2 \geqslant k$.

We take just k choices from them, and for any (S_i, S_j) among them, we take a vertex w from C and add edges between w and $S_i \cup S_j$ above. Here every vertex in C is used for and only for one time. For any (S_i, S_j) we choose above, we also add edges in $\{(u,v) \mid u \in S_i, v \in S_j\}$. Let the set of all edges added here be E_3. Let $E(G) = E_1 \cup E_2 \cup E_3$.

It is not difficult to know that the order of graph G is

$[2f(k)+1]F_v(k,k;k+1) + 3F_v(k+1, k+1; k+2) + k + 2f(k)$.

Now we will prove that graph G is K_{2k+2}-free.

If there is a complete subgraph C_0 of order $2k+2$ in G, from the construction of G we know $V(C_0)$ must be in $V(G_1) \cup V(G_2) \cup V(C)$.

Suppose $x_1 = |V(C_0) \cap V(C)|$, $x_2 = |V(C_0) \cap V(G_1)|$, and $x_3 = |V(C_0) \cap V(G_2)|$.

We know the clique numbers of C, G_1 and G_2 are all k. Therefore we have $x_i \geqslant 2$ for any $i \in \{1, 2, 3\}$.

Let $u \in V(C_0) \cap V(G_1)$ and $v \in V(C_0) \cap V(G_2)$. From the construction of G, we know that there are at most 3 common neighbors of u and v in $V(C)$. So $2 \leqslant x_1 \leqslant 3$. If $x_1 = 2$, then $x_2 = x_3 = k$, and we know it is impossible from the construction of G. If $x_1 = 3$, then from $3 + k - 1 + k - 1 = 2k + 1 < 2k + 2$ we know that either x_2 or x_3 equals k. We might suppose $x_2 = k$ as well. From the construction of G, $x_1 = 3$ and $x_2 = k$, we have $x_3 = 0$, which contradicts with $x_3 \geq 2$.

From all above we have graph G is K_{2k+2}-free.

Now, we give G any red-blue vertex coloring, i.e., we color every vertex in G with red or blue. To finish the proof, we need only to prove $G \to (2k+1, 2k+1)^v$ now.

Suppose there is not monochromatic complete subgraph of order $2k+1$ in G.

From $S \in \mathcal{F}_v(k,k;k+1)$ and S_i is isomorphic to S we know $S_i \to (k,k)^v$ for any i satisfies $0 \leq i \leq 4f(k)$. Similarly we have $T_t \to (k+1,k+1)^v$ for any $t \in \{1,2,3\}$.

Because $S_0 \to (k,k)^v$, we may suppose there is red K_k in S_0 as well. From the construction of graph G, and there is not monochromatic K_{2k+1} in G, we have there is blue K_{k+1} in T_t for any $t \in \{1,2,3\}$. So there is red K_k in S_i for any i satisfies $0 \leq i \leq 4f(k)$.

Since there is blue K_{k+1} in T_3, we know there is at least one vertex of color red in C.

Suppose this vertex be w in C, and its set of neighbors in $V(G_1) \cup V(G_2)$ is $V(S_i) \cup V(S_j)$. From the construction of graph G we know the subgraph induced by $V(S_i) \cup V(S_j) \cup \{w\}$ must contain red K_{2k+1}. Therefore $G \to (2k+1, 2k+1)^v$.

4 New upper bound for vertex Folkman number $F_v(k,k;k+1)$

Theorem 7 Suppose $0 < r < \frac{1}{2}\log_2 3 - \frac{3}{4}$, for any $k \geq a_0 = \left\lceil \left(\frac{7}{2^{\frac{1}{4}-r} - \frac{2\sqrt{3}}{3}}\right)^2 \right\rceil$ we have $F_v(k,k;k+1) \leq c_0(k-1)^{\frac{1}{4}\log_2(k-1)-r}$, in which $c_0 = \max\{\frac{F_v(i,i;i+1)}{(i-1)^{\frac{1}{4}\log_2(i-1)-r}} \mid a_0 \leq i \leq 2a_0\}$.

Proof It is not difficult to see that $F_v(k,k;k+1) \leq c_0(k-1)^{\frac{1}{4}\log_2(k-1)-r}$ if $a_0 \leq k \leq 2a_0$. Now we will prove the inequality for $2a_0 \leq k \leq 4a_0$.

From $a_0 = \left\lceil \left(\frac{7}{2^{\frac{1}{4}-r} - \frac{2\sqrt{3}}{3}}\right)^2 \right\rceil$ and $0 < r < \frac{1}{2}\log_2^3 - \frac{3}{4}$, we have

$$a_0 = \left\lceil \left(\frac{7}{2^{\frac{1}{4}-r} - \frac{2\sqrt{3}}{3}}\right)^2 \right\rceil \geq \left\lceil \left(\frac{7}{2^{\frac{1}{4}-r}}\right)^2 \right\rceil \geq \left\lceil \left(\frac{7}{2^{\frac{1}{4}}}\right)^2 \right\rceil = \left\lceil \frac{49}{\sqrt{2}} \right\rceil \geq 25.$$

Therefore we have $a_0 \geq 25$. Suppose $k \geq a_0$. So $k \geq 25$.

Case 1 For $2a_0 + 1 \leq 2k \leq 4a_0$, we have

$F_v(2k, 2k; 2k+1) \leq 5F_v(k,k;k+1) \leq 5c_0(k-1)^{\frac{1}{4}\log_2(k-1)-r}$.

From $k \geq 25$ and $0 < r < \frac{1}{2}\log_2 3 - \frac{3}{4}$ we have $2^{\frac{1}{4}-r} \cdot \sqrt{k-1} > 2^{\frac{1}{4}-(\frac{1}{2}\log_2 3 - \frac{3}{4})}\sqrt{24} = 2^{1-\frac{1}{2}\log_2 3}\sqrt{24} = \frac{2}{\sqrt{3}}\sqrt{24} = 4\sqrt{2} > 5.$

From $c_0(2k-1)^{\frac{1}{4}\log_2(2k-1)-r} > c_0(2k-2)^{\frac{1}{4}\log_2(2k-2)-r} = 2^{\frac{1}{4}-r}c_0\sqrt{k-1}(k-1)^{\frac{1}{4}\log_2(k-1)-r}$ and $2^{\frac{1}{4}-r}\sqrt{k-1} > 5$ we have $F_v(2k, 2k; 2k+1) \leq c_0(2k-1)^{\frac{1}{4}\log_2(2k-1)-r}$.

Case 2 For $2a_0 + 1 \leq 2k+1 \leq 4a_0$, we have $a_0 \leq k \leq 2a_0 - 1$.

From $k \geq 25$ and $f(k) = \left\lceil \sqrt{\frac{k}{3}} \right\rceil$, we have

$2f(k) = 2\left\lceil \sqrt{\frac{k}{3}} \right\rceil < 2\sqrt{\frac{k}{3}} + 2 < 3\sqrt{\frac{k}{3}} = \sqrt{3k} < k$.

Because $2f(k) < k$, $F_v(k,k;k+1) \geq 2k$ and $(k-1)^{\frac{1}{4}\log_2(k-1)-r} \leq k^{\frac{1}{4}\log_2 k-r}$, from theorem 6 we have

$F_v(2k+1, 2k+1; 2k+2) \leq [2f(k)+1]F_v(k,k;k+1)k + 3F_v(k+1,k+1;k+2) + k + 2f(k)$,

$F_v(2k+1, 2k+1; 2k+2) \leq [2f(k)+1]F_v(k,k;k+1)k + 3F_v(k+1,k+1;k+2) + k + k$,

$F_v(2k+1, 2k+1; 2k+2) \leq [2f(k)+2]F_v(k,k;k+1)k + 3F_v(k+1,k+1;k+2)$,

$F_v(2k+1, 2k+1; 2k+2) \leq [2f(k)+2]c_0(k-1)^{\frac{1}{4}\log_2(k-1)-r} + 3c_0k^{\frac{1}{4}\log_2 k-r}$,

$F_v(2k+1, 2k+1; 2k+2) \leq [2f((k)+5]c_0k^{\frac{1}{4}\log_2 k-r}$.

From $k \geq a_0 = \left\lceil \left(\frac{7}{2^{\frac{1}{4}-r} - \frac{2\sqrt{3}}{3}}\right)^2 \right\rceil$ we have $\sqrt{k} \geq \frac{7}{2^{\frac{1}{4}-r} - \frac{2\sqrt{3}}{3}}$. So $\left(2^{\frac{1}{4}-r} - \frac{2\sqrt{3}}{3}\right)\sqrt{k} \geq 7$, and we have

$2^{\frac{1}{4}-r}\sqrt{k} \geq \frac{2\sqrt{3}}{3}\sqrt{k} + 7 = 2\sqrt{\frac{k}{3}} + 7 > 2\left\lceil \sqrt{\frac{k}{3}} \right\rceil + 5.$

From $2^{\frac{1}{4}-r}\sqrt{k} > 2\left\lceil \sqrt{\frac{k}{3}} \right\rceil + 5 = 2f(k) + 5$ and $c_0(2k)^{\frac{1}{4}\log_2(2k)-r} = 2^{\frac{1}{4}-r}c_0\sqrt{k}\,k^{\frac{1}{4}\log_2 k-r}$ we have

$$F_v(2k+1, 2k+1; 2k+2) \leqslant c_0(2k)^{\frac{1}{4}\log_2(2k)+1}.$$

We have proved $F_v(k, k; k+1) \leqslant c_0(k-1)^{\frac{1}{4}\log_2(k-1)-r}$ for $2a_0 \leqslant k \leqslant 4a_0$. We can prove the inequality for any $k \geqslant a_0$ by induction similarly.

From all above we complete the proof.

We may give theorem 7 a simple form as following.

Theorem 8 For any r satisfies $0 < r < \frac{1}{2}\log_2 3 - \frac{3}{4}$, there are $N(r) > 0$ and $c(r) > 0$ such that

$$F_v(k, k; k+1) \leqslant c(r)(k-1)^{\frac{1}{4}\log_2(k-1)-r}.$$

for any $k \geqslant N(r)$, in which both $N(r)$ and $c(r)$ are constants only depending on r.

Remark Theorem 5 in this paper was also proven in reference[9] independently.

An earlier version of this paper was submitted to Journal of Graph Theory in Dec 2006, and was rejected at Feb 2008, mainly for the results in reference[10] gotten by non-constructive methods are much better than theorem 8 in this paper. Thanks to the referees who gave many advices on improving the writing of that version. Even so, we submit this paper here after some necessary changes, mainly for our methods are constructive. The inequalities proven in this paper can be used to give upper bounds for vertex Folkman numbers, in particular, those small ones.

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(上接第 210页 Continue from page 210)

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