广西科学 Guangxi Sciences 2007,14(4):381～385

Dynamics of the Difference Equation $x_{n+1}=(\alpha+B_1x_{n-1}+B_3x_{n-3}+\cdots+B_{2k+1}x_{n-2k-1})/(A+B_0x_n+B_2x_{n-2}+\cdots+B_{2k}x_{n-2k})$ *

差分方程 $x_{n+1}=(\alpha+B_1x_{n-1}+B_3x_{n-3}+\cdots+B_{2k+1}x_{n-2k-1})/(A+B_0x_n+B_2x_{n-2}+\cdots+B_{2k}x_{n-2k})$的动力学

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Abstract: Dynamics of the following difference equation will be investigated: $x_{n+1}=(\alpha+B_1x_{n-1}+B_3x_{n-3}+\cdots+B_{2k+1}x_{n-2k-1})/(A+B_0x_n+B_2x_{n-2}+\cdots+B_{2k}x_{n-2k})$, $n=0,1,\cdots$ the nature of the soultion of the difference equation will be investigated in four cases.

Key words: difference equation, boundedness, periodic solution, global attractor

摘要:考察差分方程 $x_{n+1}=(\alpha+B_1x_{n-1}+B_3x_{n-3}+\cdots+B_{2k+1}x_{n-2k-1})/(A+B_0x_n+B_2x_{n-2}+\cdots+B_{2k}x_{n-2k})$, $n=0,1,\cdots$ 的动力学行为,在 4 种情形下分别讨论方程解的性质.

关键词:差分方程 有界性 周期解 全局吸因子

中图法分类号:O241.3 **文献标识码**:A **文章编号**:1005-9164(2007)04-0381-05

1 Introduction

In Reference [1], the following difference equation was investigated,

$$x_{n+1}=(\alpha+B_1x_{n-1}+B_3x_{n-3}+\cdots+B_{2k+1}x_{n-2k-1})/(A+B_0x_n+B_2x_{n-2}+\cdots+B_{2k}x_{n-2k}),\quad n=0,1,\cdots \tag{1.1}$$

where k is a non-negative integer, the parameters $\alpha, A, B_i, i=0,1,2,\cdots,2k+1$ are non-negative real numbers, the initial conditions $x_{-2k-1}, x_{-2k}, \cdots, x_{-1}, x_0$ are arbitrary non-negative real numbers such that the denominator of Equation (1.1) is never zero, and $pq>0$, where $q=\sum_{i=1}^{k+1}B_{2i-1}$, $p=\sum_{i=0}^{k}B_{2i}$.

In certain conditions, Equation (1.1) exhibits the following period-two trichotomy characters.

(1) Every solution of Equation (1.1) has a finite limit if and only if $q<A$.

(2) Every solution of Equation (1.1) converges to a period-two solution of Equation (1.1) if and only if $q=A$.

(3) There exists unbounded solution of Equation (1.1) if and only if $q>A$.

In this paper, the case $pq=0$ of Equation (1.1) is investigated, and the following results are established.

Theorem 1.1 Suppose that $\alpha>0, q=0, p>0, A=0$. Then the positive equilibrium $\bar{x}$ of Equation

收稿日期:2006-12-27

修回日期:2007-06-25

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* Supported by the Natural Science Foundation of China (10661001) and the Research Fund of Technology Master of Guangxi University of Technology(070232).

(1.1) is a non-hyperbolic equilibrium point. If there exists $t \in \{0,1,2,\cdots,k\}$ such that $B_{2t} > 0$ and $B_{2i} = 0$ for all $i \in \{0,1,2,\cdots,k\} - \{t\}$, then every positive solution of Equation (1.1) is a $4t+2$ period solution of Equation (1.1).

Theorem 1.2 Suppose that $\alpha > 0$, $q = 0$, $p > 0$, $A > 0$. Then the positive equilibrium point $\bar{x} = \dfrac{-A + \sqrt{A^2 + 4p\alpha}}{2p}$ of Equation (1.1) is globally asymptotically stable.

Theorem 1.3 Suppose that $\alpha = 0, q > 0, p = 0, A > 0$.

(1) Suppose $q > A$, then every positive solution of Equation (1.1) converges to $+\infty$.

(2) Suppose $q < A$, then every positive solution of Equation (1.1) converges to 0.

(3) Suppose $q = A$ and Equation (1.1) satisfies the following hypotheses: for every $t \in \{0,1,\cdots,r-1\}$, there exist $i \in X$ and $s \in \{0,1,\cdots,k+1\}$ such that $t = i - sr$, where $X = \{i \mid B_{2i-1} \neq 0, i = 1,2,\cdots,k+1\}$ and $r = \min X$. Then every non-negative solution of Equation (1.1) converges to a period-two solution of Equation (1.1).

Theorem 1.4 Suppose that $\alpha > 0, q > 0, p = 0, A > 0$.

(1) Suppose $q \geqslant A$, then every non-negative solution of Equation (1.1) converges to $+\infty$.

(2) Suppose $q < A$, then every non-negative solution of Equation (1.1) converges to $\dfrac{\alpha}{A-q}$.

2 Preliminaries

Now some definitions and the known results which are employed in the investigation are listed.

Let $f: J^{k+1} \to J$ be a continuous function, where k is a non-negative integer and J is an interval of real numbers. Consider the difference equation

$$y_{n+1} = f(y_n, y_{n-1}, \cdots, y_{n-k}), n = 0,1,\cdots \quad (2.1)$$

with initial conditions $y_{-k}, y_{-k+1}, \cdots, y_0 \in J$.

$\bar{y}$ is an equilibrium point of Equation (2.1) if $f(\bar{y}, \bar{y}, \cdots, \bar{y}) = \bar{y}$.

We now impose the further restriction that the function $f(u_0, u_1, \cdots, u_k)$ be continuously differentiable.

The linearized equation of Equation (2.1) about the equilibrium point $\bar{y}$ is the linear difference equation

$$Z_{n+1} = a_0 Z_n + a_1 Z_{n-1} + \cdots + a_k Z_{n-k}, n = 0,1,\cdots \quad (2.2)$$

where for each $i = 0,1,\cdots,k$

$$a_i = \frac{\partial f}{\partial u_i}(\bar{y}, \bar{y}, \cdots, \bar{y}).$$

The characteristic equation of Equation (2.2) is the equation

$$\lambda^{k+1} - a_0\lambda^k - a_1\lambda^{k-1} - \cdots - a_{k-1}\lambda - a_k = 0. \quad (2.3)$$

Definition 2.1[2,3] Let $\bar{y}$ be an equilibrium point of Equation (2.1),

(a) $\bar{y}$ is called locally stable if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $y_{-k}, y_{-k+1}, \cdots, y_{-1}, y_0 \in J$ and

$$\sum_{i=-k}^{0} |y_i - \bar{y}| < \delta,$$

then

$$|y_n - \bar{y}| < \varepsilon \text{ for all } n \geqslant -k.$$

(b) $\bar{y}$ is called locally asymptotically stable if it is locally stable and if there exists $\gamma > 0$ such that if $y_{-k}, y_{-k+1}, \cdots, y_{-1}, y_0 \in J$ and

$$\sum_{i=-k}^{0} |y_i - \bar{y}| < \gamma,$$

then

$$\lim_{n\to\infty} y_n = \bar{y}.$$

(c) $\bar{y}$ is called a global attractor if, for every $y_{-k}, y_{-k+1}, \cdots, y_{-1}, y_0 \in J$, we have

$$\lim_{n\to\infty} y_n = \bar{y}.$$

(d) $\bar{y}$ is called globally asymptotically stable if it is locally stable and a global attractor.

(e) $\bar{y}$ is called hyperbolic if no root of Equation (2.3) has modulus equal to one. Otherwise it is called non-hyperbolic.

The following result is useful in determining the local stability character of the equilibrium point $\bar{y}$ of Equation (2.1).

Theorem 2.1 (The Linearized Stability Theorem)[4]

If every root of Equation (2.3) has absolute value less than one, then the equilibrium point $\bar{y}$ of Equation (2.1) is locally asymptotically stable.

Theorem 2.2 (Clark's Theorem)[5]

Assume that

$$\sum_{i=0}^{k} |a_i| < 1,$$

then every root of Equation (2.3) has absolute value less than one.

3 The proof of main results

Suppose that $\alpha > 0, q = 0, p > 0, A = 0$. Then the equilibrium point of Equation (1.1) is $\bar{x} = \sqrt{\frac{\alpha}{p}}$.

Suppose that $\bar{x} > 0$ is the equilibrium point of Equation (1.1). The linearized equation of Equation (1.1) with respect to $\bar{x}$ is

$$Z_{n+1} + \frac{1}{p}(B_0 Z_n + B_2 Z_{n-2} + \cdots + B_{2k} Z_{n-2k}) = 0,$$

with characteristic equation

$$\lambda^{2k+2} + \frac{1}{p}(B_0 \lambda^{2k+1} + B_2 \lambda^{2k-1} + \cdots + B_{2k}\lambda) = 0. \quad (3.1)$$

Proof of theorem 1.1 It is easy to see that -1 is one root of Equation (3.1). Hence the positive equilibrium $\bar{x}$ of Equation (1.1) is a non-hyperbolic equilibrium point.

Now, suppose that there exists $t \in \{0,1,2,\cdots,k\}$ such that $B_{2t} > 0$ and $B_{2i} = 0$ for all $i \in \{0,1,2,\cdots,k\} - \{t\}$. Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a positive solution of Equation(1.1), then

$$x_{n+1} = \frac{\alpha}{B_{2t} x_{n-2t}}, x_{n+2t+2} = \frac{\alpha}{B_{2t} x_{n+1}}.$$

It follows that

$$x_{n+2t+2} = x_{n-2t} \text{ for all } n = 0,1,\cdots,$$

namely, $\{x_n\}_{n=-2k-1}^{\infty}$ is a $4t+2$ period solution of equation (1.1). The proof of Theorem 1.1 is completed.

Suppose that $\alpha > 0, q = 0, p > 0, A > 0$. Then the equilibrium point of Equation (1.1) are the positive solution of the equation

$$p\bar{x}^2 + A\bar{x} - \alpha = 0.$$

Suppose that $\bar{x} > 0$ is the equilibrium point of Equation (1.1). The linearized equation of Equation (1.1) with respect to $\bar{x}$ is

$$Z_{n+1} + \frac{\bar{x}}{A + p\bar{x}}(B_0 Z_n + B_2 Z_{n-2} + \cdots + B_{2k} Z_{n-2k}) = 0,$$

with characteristic equation

$$\lambda^{2k+2} + \frac{\bar{x}}{A + p\bar{x}}(B_0 \lambda^{2k+1} + B_2 \lambda^{2k-1} + \cdots + B_{2k}\lambda) = 0. \quad (3.2)$$

Proof of theorem 1.2 By Theorems 2.1, 2.2 and Equation (3.2), it is easy to see that $\bar{x} > 0$ is locally asymptotically stable equilibrium point of Equation (1.1).

Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a positive solution of Equation (1.1). It is obvious that

$$\max\{\frac{\alpha}{A}, x_{-2k}, x_{-2k+1}, \cdots, x_{-1}, x_0\}$$

is its upper bounded. Set

$$I = \liminf_{n\to\infty} x_n \text{ and } S = \limsup_{n\to\infty} x_n,$$

from Equation (1.1) we can get

$$S \leqslant \frac{\alpha}{A + pI} \leqslant I \leqslant S.$$

Hence

$$I = S,$$

and

$$I = S = \frac{-A + \sqrt{A^2 + 4p\alpha}}{2p}.$$

The proof of Theorem 1.2 is completed.

Proof of theorem 1.3 (1) Suppose $q > A$. Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a positive solution of Equation(1.1) and set

$$m = \min\{x_{-2k-1}, x_{-2k}, \cdots, x_{-1}, x_0\}, r = \frac{q}{A},$$

then

$$x_1 = (B_1 x_{-1} + B_3 x_{-3} + \cdots + B_{2k+1} x_{-2k-1})/A \geqslant \frac{qm}{A} = rm,$$

$$x_2 = (B_1 x_0 + B_3 x_{-2} + \cdots + B_{2k+1} x_{-2k})/A \geqslant \frac{qM}{A} = rm,$$

$$\vdots$$

$$x_{2k+2} = (B_1 x_{2k} + B_3 x_{2k-2} + \cdots + B_{2k+1} x_0)/A \geqslant \frac{qM}{A} = rm,$$

$$x_{2k+3} = (B_1 x_{2k+1} + B_3 x_{2k-1} + \cdots + B_{2k+1} x_1)/A \geqslant \frac{q\frac{qM}{A}}{A} = r^2 m,$$

$$\vdots$$

$$x_{4(k+1)} = (B_1 x_{4k+3} + B_3 x_{4k+1} + \cdots + B_{2k+1} x_{2k+3})/A \geqslant \frac{q\frac{qM}{A}}{A} = r^2 m.$$

It follows by induction that for all $n \geqslant 1$, we have

$$x_{2(n-1)(k+1)+1}, x_{2(n-1)(k+1)+2}, \cdots, x_{2n(k+1)} \geqslant r^n m.$$

Note that $r > 1$, we have

$$\lim_{n\to\infty} x_n = \infty,$$

and the proof is completed.

(2) Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a positive solution of Equation (1.1) and set

$$M = \max\{x_{-2k-1}, x_{-2k}, \cdots, x_{-1}, x_0\},$$

then

$$x_1=(B_1x_{-1}+B_3x_{-3}+\cdots+B_{2k+1}x_{-2k-1})/A\leqslant \frac{qM}{A}<M.$$

It follows by induction that for all $n\geqslant 1$,we have $x_n\leqslant M$. Set $I=\liminf\limits_{n\to\infty} x_n$ and $S=\limsup\limits_{n\to\infty} x_n$.

By Equation (1.1) we have

$$S\leqslant \frac{q}{A}S,$$

i.e.

$$(1-\frac{q}{A})S\leqslant 0.$$

Since

$$q<A,$$

we have $S=0$. Hence

$$\lim_{n\to\infty}x_n=0,$$

and the proof is completed.

(3) Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a positive solution of Equation (1.1) and set

$$M=\max\{x_{-2k-1},x_{-2k},\cdots,x_{-1},x_0\},$$

then

$$x_1=(B_1x_{-1}+B_3x_{-3}+\cdots+B_{2k+1}x_{-2k-1})/A\leqslant \frac{qM}{A}=M.$$

It follows by induction that for all $n\geqslant 1$,we have $x_n\leqslant M$. Set $\liminf\limits_{n\to\infty}x_{2n}=m$, and $y_n=x_{2n}-m$ for all $n=-k,-k+1,\cdots,-1,0,1,\cdots$. It is obvious that

$$\liminf_{n\to\infty}y_n=0$$

and

$$y_{n+1}=(B_1y_n+B_3y_{n-1}+\cdots+B_{2k+1}y_{n-k})/A,n=0,1,\cdots.$$

Suppose

$$\lim_{j\to\infty}y_{m_j+1}=0,$$

Note that

$$y_{m_j+1}=(B_1y_{m_j}+B_3y_{m_j-1}+\cdots+B_{2k+1}y_{m_j-k})/A,j=0,1,\cdots$$

and

$$B_{2i-1}\neq 0 \text{ for all } i\in X,$$

we have

$$\lim_{j\to\infty}y_{m_j-i+1}=0 \text{ for all } i\in X.$$

It follows by induction that

$$\lim_{j\to\infty}y_{m_j+1-i-mr}=0 \qquad (3.3)$$

for all $i\in X$ and all $m\geqslant 0$.

Suppose that

$$\lim_{j\to\infty}y_{u_j+1}=a,$$

we also suppose, without loss generality, that $u_j\leqslant m_j-k-1$ for all $j\geqslant 0$. For $j\geqslant 0$,suppose $u_j-m_j=-n_jr-t_j$, where $n_j\geqslant 0$ and $t_j\in\{0,1,\cdots,r-1\}$.

By hypothesis we know that there exist $i_j\in X$ and $s_j\in\{0,1,\cdots,k+1\}$ such that $t_j=i_j-s_jr$, consequently $u_j+1=m_j+1-i_j-(n_j-s_j)r$ for all $j\geqslant 0$.

Note that $n_j\geqslant s_j$ for all $j\geqslant 0$, by Equation (3.3) we get

$$\lim_{j\to\infty}y_{u_j+1}=\lim_{j\to\infty}y_{m_j+1-i_j-(n_j-s_j)r}=0.$$

Hence

$$\lim_{n\to\infty}y_n=0,$$

and

$$\lim_{n\to\infty}x_{2n}=m.$$

Similarly, there exists an M ,such that

$$\lim_{n\to\infty}x_{2n+1}=M,$$

and the proof is completed.

The proof of Theorem 1.3 is completed.

Proof of theorem 1.4 (1) $q\geqslant A$. Let $\{x_n\}_{n=-2k-1}^{\infty}$ be a non-negative solution of Equation (1.1), then

$$x_1=(\alpha+B_1x_{-1}+B_3x_{-3}+\cdots+B_{2k+1}x_{-2k-1})/A\geqslant\frac{\alpha}{A},$$

$$x_2=(\alpha+B_1x_0+B_3x_{-2}+\cdots+B_{2k+1}x_{-2k})/A\geqslant\frac{\alpha}{A},$$

$$\vdots$$

$$x_{2k+2}=(\alpha+B_1x_{2k}+B_3x_{2k-2}+\cdots+B_{2k+1}x_0)/A\geqslant\frac{\alpha}{A},$$

$$x_{2k+3}=(\alpha+B_1x_{2k+1}+B_3x_{2k-1}+\cdots+B_{2k+1}x_1)/A\geqslant 2\frac{\alpha}{A},$$

$$\vdots$$

$$x_{4(k+1)}=(\alpha+B_1x_{4k+3}+B_3x_{4k+1}+\cdots+B_{2k+1}x_{2k+3})/A\geqslant 2\frac{\alpha}{A}.$$

It follows by induction that for all $n\geqslant 1$, we have

$$x_{2(n-1)(k+1)+1},x_{2(n-1)(k+1)+2},\cdots,x_{2n(k+1)}\geqslant n\frac{\alpha}{A}.$$

Hence we have

$$\lim_{n\to\infty}x_n=\infty,$$

and the proof is completed.

(2) Set

$$x_n=y_n+\frac{\alpha}{A-q},n=0,1,\cdots,$$

from Theorem 1.3 (2) we know that $\{y_n\}_{n=-2k-1}^{\infty}$

converges to 0, hence $\{x_n\}_{n=-2k-1}^{\infty}$ converges to $\frac{\alpha}{A-q}$.

The proof of Theorem 1.4 is completed.

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表 1 中的相关符号意义为:Problem 是测试问题的名称;Dim 是目标函数的维数;NI 是算法迭代的次数;NF 是函数值计算的次数;NG 是函数梯度计算的次数;PRPSWP 是(0.8)式＋SWP;DYHybrid 是(0.17)式＋WWP;NewHybrid 是(1.1)式＋WWP.

表 1 的数据结果显示,对测试问题集的 54 个目标函数,PRPSWP、DYHybrid 和 NewHybrid 方法求解失败的个数分别为 9、7、6 个. NewHybrid 方法的 NI/NF/NG 数据优于 DYHybrid 和 PRPSWP 方法.

致谢:

本文在完成过程中得到广西大学韦增欣教授的悉心指导,他对本文提出宝贵的修改意见,我们谨此深表感谢!

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