广西科学 Guangxi Sciences 2007,14(4):354~356

$\aleph_0$-Weak Base and Related Matters*

$\aleph_0$-弱基及相关问题

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Abstract: It is proved that the followings are equivalent for a space X: (1) X has a σ-discrete $\aleph_0$-weak base; (2) X has a σ-locally finite $\aleph_0$-weak base; (3) X is a $\aleph_0$-weakly first-countable and $\aleph$-space, space with $\aleph_0$-weak base is open, closed hereditary, and point-countable $\aleph_0$-weak base is cs*-network. Some relations are discussed among $\aleph_0$-weak base, sn-network, cs-network and cs*-network.

Key words: $\aleph_0$-weak base, $\aleph$-space, sn-metric space, sn-first countable, cs*-network, cs-network

摘要：证明在空间X中下列论述等价：(1)X有σ-离散的$\aleph_0$-弱基；(2)X有σ-局部有限的$\aleph_0$-弱基；(3)X是$\aleph_0$-弱第一可数的$\aleph$空间，$\aleph_0$-弱基是开、闭遗传的，点可数$\aleph_0$-弱基是cs*-网．并讨论$\aleph_0$-弱基，sn-网，cs-网以及cs*-网的关系．

关键词：$\aleph_0$-弱基　$\aleph$-空间　sn-度量空间　sn-第一可数　cs*-网　cs-网
中图法分类号：O189.1　　**文献标识码**：A　　**文章编号**：1005-9164(2007)04-0354-03

1 Definitions and preliminaries

The concept of weak base was introduced by Arhangel'skill[1]. R. Sirois-Dumais[2] defined $\aleph_0$-weak base. From their definitions it is easy to get weak base that is an $\aleph_0$-weak base. but $\aleph_0$-weak base may not be weak base. C. Liu and S. Lin[3] characterized a space X is a quotient, countable-to-one image of a metric space if and only if X has a point-countable $\aleph_0$-weak base. It is natural to ask whether there are any others characterizations about $\aleph_0$-weak base.

In this paper, all spaces are regular T_1, all maps are continuous and onto, and $\mathbf{N}$ is the set of positive integer numbers, the sequence $\{x_n : n \in \mathbf{N}\}$, the sequence $\{P_n : n \in \mathbf{N}\}$ of subsets and the sequence $\{\mathscr{P}_n : n \in \mathbf{N}\}$ of collections of subsets are abbreviated to $\{x_n\}$, $\{P_n\}$ and $\{\mathscr{P}_n\}$ respectively. For terms which are not defined here, please refer to Reference[3] and related references.

Definition 1.1[3] Let $\mathscr{B}$ be a family of subsets of a space X. $\mathscr{B}$ is said to be an $\aleph_0$-weak base for X if $\mathscr{B} = \bigcup\{\mathscr{B}_x(n) : x \in X, n \in \mathbf{N}\}$ satisfies

(1) For each $x \in X, n \in \mathbf{N}$, $\mathscr{B}_x(n)$ is closed under finite intersections and $x \in \bigcap \mathscr{B}_x(n)$.

(2) A subset U of X is open if and only if whenever $x \in U$ and $n \in \mathbf{N}$, there exists a $B_x(n) \in \mathscr{B}_x(n)$ such that $B_x(n) \in U$.

X is called $\aleph_0$-weakly first-countable or weakly quasi-first-countable in the sense of Sirois-Dumais[2] if X has $\aleph_0$-weak base $\mathscr{B} = \bigcup \mathscr{B}_x(n) : x \in X, n \in \mathbf{N}\}$, $\mathscr{B}_x(n)$ is countable for any $x \in X, n \in \mathbf{N}$.

Definition 1.2[4] Let X be a space, $P \subset X$ is called a sequential neighbourhood of x in X, if each sequence converging to x in X is eventually in P.

Definition 1.3 Let $f: X \to Y$ be a closed map if each closed subset F of X, then $f(F)$ is closed in Y.

收稿日期：2007-06-21
修回日期：2007-07-25
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* Supported by the Natural Science Foundation of Guangxi (No. 0728035).

Definition 1.4 Let $\mathscr{P}$ be a cover of X, Then $\mathscr{P}$ is called a k-network for X if for any compact set K and for any open set U such that $K \subset \bigcup \mathscr{P}' \subset U$ for some finite $\mathscr{P}' \subset \mathscr{P}$.

2 Main results

In this section, we give a characterization about $\aleph_0$-weak base, and some related results.

Lemma 2.1[2] Every $\aleph_0$-weakly first-countable space is sequential.

Lemma 2.2[3] X has a point-countable $\aleph_0$-weak base $\mathscr{B} = \bigcup\{\mathscr{B}_x(n): x \in X, n \in \mathbf{N}\}$. L be a sequence converging to $x \notin L$ in X. Then there exists a subsequence L' of L and $n_0 \in \mathbf{N}$ such that L' is eventually in $B_x(n_o, m)$ for any $m \in \mathbf{N}$.

Lemma 2.3 $\mathscr{B}$ is a point-countable $\aleph_0$-weak base, then $\mathscr{B}$ is a cs*-network.

Proof Let $\mathscr{B} = \bigcup\{\mathscr{B}_x(n): x \in X, n \in \mathbf{N}\}$ is a $\aleph_0$-weak base for X, here each $\mathscr{B}_x(n) = \{B_x(n,m): m \in \mathbf{N}\}$ with $B_x(n, m+1) \subset B_x(n,m)$ for each $m \in \mathbf{N}$. Then Let L be a sequence converging to x in X and U is an open set of $X, L \subset U$ and $x \in U$. By the Lemma 2.2, there exists a subsequence L' of L and $n_0 \in \mathbf{N}$ such that L' eventually in $B_x(n_0, m)$ for any $m \in \mathbf{N}$. From the definition 1.1, we know that there exists $m_0 \in \mathbf{N}$ such that $B_x(n_0, m_0) \subset U$. So the $\mathscr{B}$ is a cs*-network.

Theorem 2.4 Let $\mathscr{B} = \bigcup\{\mathscr{B}_x(n): x \in X, n \in \mathbf{N}\}$ be an $\aleph_0$-weak base of a space X and A a closed subset of X. Then $\{A \cap B: B \in \mathscr{B}\}$ is an $\aleph_0$-weak base of A.

Proof For $x \in A$. It is easy to see that $\{A \cap B: B \in \mathscr{B}\}$ is closed under finite intersections and $x \in (\bigcap \mathscr{B}_x(n)) \cap A$.

If U is open in A, there is an open set V of X such that $U = V \cap A$. For $x \in U$, there is a $B \in \mathscr{B}_x(n)$ such that $x \in B \subset V$. $x \in B \cap A \in \{B \cap A: B \in \mathscr{B}_x(n)\}$. If for each $x \in U$, there exists a $B \cap A \in \{B \cap A: B \in \mathscr{B}_x(n)\}$ such that $x \in B \cap A \subset U$. We prove that U is open in A. For $x \in V = X \backslash (A \backslash U)$. If $x \in (X \backslash A)$, since A is closed, there is a $B \in \mathscr{B}_x(n)$ such that $B \subset V$. If $x \in U$, there is a $B \cap A \in \{B \cap A: B \in \mathscr{B}_x(n)\}$ such that $x \in B \cap A \subset U$, it means that $B \in \mathscr{B}_x(n)$ and $B \subset V$. Hence V is open in X, therefore $U = V \cap A$ is open in A.

Theorem 2.5 Let $\mathscr{B} = \bigcup\{\mathscr{B}_x(n): x \in X, n \in \mathbf{N}\}$ be an $\aleph_0$-weak base of a space X and A an open subset of X. Then $\{A \cap B: B \in \mathscr{B}\}$ is an $\aleph_0$-weak base of A.

Proof This proof is similar to Theorem 2.4.

From Theorems 2.4 and 2.5, it is easy to get $\aleph_0$-weak base is open, closed hereditary.

Lemma 2.6[5] Let $\mathscr{P}$ be a σ-hereditarily closure-preserving collection of subsets of a space X. If $\mathscr{P}$ is a cs*-network, then $\mathscr{P}$ is a k-network of X.

Theorem 2.7 The following are equivalent for a space X.

(1) X has a σ-discrete $\aleph_0$-weak base;

(2) X has a σ-locally finite $\aleph_0$-weak base;

(3) X is a $\aleph_0$-weakly first-countable and $\aleph$-space.

Proof (1)→(2) obviously.

We prove that (2)→(3). Let $\mathscr{B}$ be a σ-locally finite $\aleph_0$-weak base, then $\mathscr{B}$ is a point-countable $\aleph_0$-weak base, X is a $\aleph_0$-weakly first-countable space, from Lemma 2.3, $\mathscr{B}$ is a cs*-network. $\mathscr{B}$ be a σ-locally finite $\aleph_0$-weak base, then $\mathscr{B}$ is a σ-hereditarily closure-preserving point-countable $\aleph_0$-weak base. From Lemma 2.6 X has σ-hereditarily closure-preserving k-network. So X is a $\aleph$-space.

Now we prove (3)→(1).

Since X is an $\aleph$-space, by Theorem 4 in[6], we can assume that X has a σ-discrete cs-network $\mathscr{P}$, where $\mathscr{P}$ is closed under finite intersections. Let $\bigcup\{\mathscr{B}_x(n): x \in X, n \in \mathbf{N}\}$ be an $\aleph_0$-weak base of X, then for each $x \in X, n \in \mathbf{N}, \mathscr{B}_x(n)$ is countable, here each $\mathscr{B}_x(n) = \{B_x(n,m): m \in \mathbf{N}\}$ with $B_x(n, m+1) \subset B_x(n,m)$ for each $m \in \mathbf{N}$. For each $n \in \mathbf{N}$, let $\mathscr{P}_x(n) = \{P \in \mathscr{P}: B_x(n,m) \subset P$ for some $m \in \mathbf{N}\}$. Then $\mathscr{P}_x(n)$ is closed under finite intersections. $\bigcup\{\mathscr{P}_x(n): x \in X, n \in \mathbf{N}\}$ is an σ-discrete collection.

$\mathscr{P}_x(n)$ is a network of x in X. Suppose not, there is a neighbourhood U of x in X such that $P \not\subset U$ for each $P \in \mathscr{P}_x(n)$. Let $P \in \mathscr{P}: x \in P \subset U\} = \{P_k: k \in \mathbf{N}\}$. Then $B(n,m) \not\subset P_k$ for any $m, k \in \mathbf{N}$. Pick $x_{mk} \in B(n,m) \backslash P_k$ for each $m \geqslant k$. Let $y_i = x_{mk}$, where $i = k + m(m-1)/2$. Then the sequence y_i converges to x in X because $\{B_x(n,m): m \in \mathbf{N}\}$ is a decreasing network of x in X. Since $\mathscr{P}$ is a cs-network of X, there exists $k, j \in \mathbf{N}$ such that $\{y_i: i \geqslant j\} \subset P_k$. Pick $i \geqslant j$ such that $y_i = x_{mk}$ for some $m \geqslant k$, then $x_{mk} \in P_k$, a contradiction.

Put $\mathscr{B} = \bigcup\{P_x(n): x \in X, n \in \mathbf{N}\}$. We shall prove that $\mathscr{B}$ is an $\aleph_0$-weak base. We only need to prove that a subset V of X is open if whenever $x \in V, n \in \mathbf{N}$, there exists a $P_x(n) \in \mathscr{P}_x(n)$ such that $P_x(n) \subset V$. If V is not open in X, from Lemma2. 1, we know V is not sequentially open. There is a sequence L in $X \backslash V$ converging to a point $x \in V$. By the claim in Lemma 2. 2, there exists a subsequence L' of L and $n_0 \in \mathbf{N}$ such that L' is eventually in $B_x(n_0, m)$ for any $m \in \mathbf{N}$. But $B_x(n_0, m) \subset P_x(n_0)$ for some $m \in \mathbf{N}$, L' is eventually in $P_x(n_0) \subset V$, a contradiction. So $\mathscr{B}$ is an $\aleph_0$-weak base for X.

From Theorem 2. 7, it is natural to ask the following question:

Question 2. 8 Is a space X with σ-hereditarily closure-preserving $\aleph_0$-weak base $\aleph_0$-weakly first-countable space and $\aleph$-space?

This question can be partially answered if X has σ-hereditarily closure-preserving point-countable $\aleph_0$-weak base, then X is a $\aleph_0$-weakly first-countable space and $\aleph$-space.

Lemma 2. 9[7] Let $f: X \rightarrow Y$ be a closed map and X has σ-hereditarily closure-preserving k-network, then Y is sn-metric space if and only if Y is sn-first countable space.

Theorem 2. 10 Let $f: X \rightarrow Y$ be a closed map and X has σ-hereditarily closure-preserving point-countable $\aleph_0$-weak base, then Y is sn-metric space if and only if Y is sn-first countable space.

Proof If Y is sn-metric space, Y is sn-first countable space.

If X has σ-hereditarily closure-preserving point-countable $\aleph_0$-weak base $\mathscr{B}$ From Lemma 2. 3, we get that $\mathscr{B}$ is a cs*-network. From Lemma 2. 6, X has σ-hereditarily closure-preserving k-network. From Lemma 2. 9, it is easy to get Y is an sn-metric space.

Corollary 2. 11 A space X with σ-hereditarily closure-preserving point-countable $\aleph_0$-weak base, then X is a sn-metric space if and only if X is a sn-first countable space.

Corollary 2. 12 A space X with σ-hereditarily closure-preserving point-countable $\aleph_0$-weak base, then X is an $\aleph$-space if and only if X contains no closed copy of S_{ω_1}.

In the following, we will give some remarks on $\aleph_0$-weak base, sn-network, cs*-network, cs-network.

Remark 2. 13 The $\aleph_0$-weak base may not be sn-network. If not, point-countable $\aleph_0$-weak base is point-countable sn-network. The point-countable $\aleph_0$-weak base space is sequential space, then the point-countable sn-network is point-countable weak base. This is not true.

Remark 2. 14 The $\aleph_0$-weak base may not be cs-network. Every quotient, finite-to-one image of a locally compact metric space does not have a point-countable cs-network[8]. But C. Liu and S. Lin proved that it had point-countable $\aleph_0$-weak base[3].

Acknowledgement:

The author would like to thank Zhang-yong Cai for his suggestion of proving of Theorem 2. 7.

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