Positivity for an Evolution p-Laplacian Equation*

一类发展 p-Laplacian 方程的正解

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Abstract: The weak solution of the initial boundary value problem of evolution p-Laplacian equation $\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}(|\frac{\partial u^m}{\partial x}|^{p-2}\frac{\partial u^m}{\partial x})$ is discussed. It is proved that the weak solution is positive when $m(p-1)>1$ and t is large enough.

Key words: p-Laplacian equation, evolution equation, positivity.

摘要:讨论一类发展 p-Laplacian 方程 $\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}(|\frac{\partial u^m}{\partial x}|^{p-2}\frac{\partial u^m}{\partial x})$ 的初边值问题的弱解,并证明 $m(p-1)>1$,而且 t 足够大时弱解是一个正解.

关键词: p-Laplacian 方程　发展方程　正解

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1 Introduction

In this paper, we consider the evolution p-Laplacian equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}(|\frac{\partial u^m}{\partial x}|^{p-2}\frac{\partial u^m}{\partial x}) \tag{1.1}$$

In Q with initial-boundary conditions

$$u(\pm 1,t)=0, \forall\, t\in(0,+\infty), \tag{1.2}$$

$$u(x,0)=u_0(x), \forall\, x\in[-1,1], \tag{1.3}$$

where $p>1$ and $m>0$ are two given real numbers, $u_0(x)$ is a nonzero nonnegative continuous function in $I=[-1,1]$, and $Q=I\times(0,+\infty)$.

Equation (1.1) appears in a number of applications to describe the evolution of diffusion processes, in particular non-Newtonian flow in a porous medium, see References [1,2].

The quasilinear equation (1.1) is degenerate if $p>2$ or singular if $1<p<2$, since the modulus of ellipticity degenerates($p>2$) or blows up ($1<p<2$) at points, where $\partial u^m/\partial x=0$, and so there is no classical solution in general. We consider its weak solutions here.

Definition 1.1 A nonnegative function $u(x,t)$ is called a weak solution of the problems(1.1)～(1.3), if the following conditions are fulfilled.

(1) $u\in L^\infty(Q)\cap C(Q)$, $\frac{\partial u^m}{\partial x}\in L^p_{loc}(Q)$,

(2) $$\int_{-1}^{1}u(x,T)\varphi(x,T)\mathrm{d}x-\int_{-1}^{1}u_0(x)\varphi(x,0)\mathrm{d}x=\int_0^T\int_{-1}^{1}(u\frac{\partial\varphi}{\partial t}-|\frac{\partial u^m}{\partial x}|^{p-2}\frac{\partial u^m}{\partial x}\frac{\partial\varphi}{\partial x})\mathrm{d}x\mathrm{d}t,$$

for all $T\in(0,+\infty)$ and all $\varphi\in C(0,T;W_0^{1,p}(I))$ with $\partial\varphi/\partial t\in L^p(Q)$.

The existence of solutions of the problems (1.1)～(1.3) was reported in Reference [3]. And $m=1$, the positivity of Equation (1.1) is discussed in Reference [4].

In this paper, our goal is to investigate the positivity of the solutions, and main results are obtained as follows.

Theorem 1.1 Let u be a weak solution of the problems (1.1)～(1.3), if $m(p-1)>1$, then there exists a time T such that $u(x,t)>0$, for all $(x,t)\in I$

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$\times (T, +\infty)$.

2 Fundamental lammas

Lemma 2.1 Let u be a weak solution of the problems (1.1) ~ (1.3), if the function $v(x,t)$ satisfies

$$\frac{\partial v}{\partial t} \geqslant \frac{\partial}{\partial x}\left(\left|\frac{\partial v^m}{\partial x}\right|^{p-2}\frac{\partial v^m}{\partial x}\right),$$

$$v(x,0) \geqslant u(x,0), \forall x \in I,$$

$$v(\pm 1,t) \geqslant u(\pm 1,t), \forall t \in (0,+\infty),$$

then we have $v(x,t) \geqslant u(x,t)$ for all $(x,t) \in Q$.

The proof is similar to that given in Reference [5].

Lemma 2.2 Let u be a weak solution of the problems (1.1)~(1.3), then we have

(1) if $m(p-1) > 1$, then

$$\frac{\partial u}{\partial t} \geqslant -\frac{u}{(m(p-1)-1)t} \qquad (2.1)$$

in the sense of distributions.

(2) if $0 < m(p-1) < 1$, then

$$\frac{\partial u}{\partial t} \leqslant \frac{u}{(1-m(p-1))t} \qquad (2.2)$$

in the sense of distributions.

Proof Denote $u_r(x,t) = ru(x, r^{m(p-1)-1}t)$, $(x,t) \in Q, r \in (\frac{1}{2},1)$. Clearly, u_r is a weak solution of Equation (1.1) with the following initial-boundary condition

$$u_r(x,0) = ru_0(x), \forall x \in I, u_r(\pm 1,t) = 0, \forall t \in (0,+\infty). \qquad (2.3)$$

Using (1.2) and (2.3), we get

$$u_r(\pm 1,t) = u(\pm 1,t) = 0, \forall t \in (0,+\infty), \qquad (2.4)$$

$$u_r(x,0) = ru_0(x) < u_0(x), \forall x \in I, \qquad (2.5)$$

and $u_r(x,t)$ satisfies $\frac{\partial u_r}{\partial t} \leqslant \frac{\partial}{\partial x}\left(\left|\frac{\partial u_r^m}{\partial x}\right|^{p-2}\frac{\partial u_r^m}{\partial x}\right)$.

Applying Lemma 2.1, we have

$$u_r(x,t) \leqslant u(x,t), \qquad (2.6)$$

for all $(x,t) \in Q$.

We consider the cases: (1) $m(p-1) > 1$, (2) $0 < m(p-1) < 1$, respectively.

(1) For $m(p-1) > 1$, let $\lambda = r^{m(p-1)-1}$, by (2.6) we know

$$\frac{u(x,\lambda t)^{m(p-1)-1} - u(x,t)^{m(p-1)-1}}{\lambda t - t} \geqslant$$

$$\frac{(\frac{1}{\lambda}-1)u(x,t)^{m(p-1)-1}}{\lambda t - t} = -\frac{1}{\lambda t}u(x,t)^{m(p-1)-1}.$$

Let $\lambda \to 1^-$, we get

$$(m(p-1)-1)\frac{\partial u}{\partial t} \geqslant -\frac{1}{t}u(x,t),$$

in the distribution, which implies that (2.1) holds.

(2) For $0 < m(p-1) < 1$, by (2.6) we have

$$\frac{u(x,\lambda t)^{1-m(p-1)} - u(x,t)^{1-m(p-1)}}{\lambda t - t} \leqslant$$

$$\frac{(\lambda-1)u(x,t)^{1-m(p-1)}}{\lambda t - t} = \frac{1}{t}u(x,t)^{1-m(p-1)}.$$

Let $\lambda \to 1^+$, we get $(1-m(p-1))\frac{\partial u}{\partial t} \leqslant \frac{1}{t}u(x,t)$, in the distribution, which implies that (2.2) holds. Thus the proof is completed.

Lemma 2.3 Let u be a weak solution of the problems (1.1)~(1.3), if $m(p-1) > 1$, then

$$\mathrm{supp}\, u(.,s) \subset \mathrm{supp}\, u(.,t)$$

for all s,t with $0 < s < t$.

The proof follows from Lemma 2.2.

3 Positivity of solutions

In this section we shall prove the theorem 1.1 and assume that $m(p-1) > 1$. In order to prove the theorem, we need the following lemmas.

Denote $s_+ = \max\{s,0\}$ for all $s \in (-\infty,+\infty)$.

Lemma 3.1 Let u be a weak solution of the problems (1.1)~(1.3) and $m(p-1) > 1$. If $u_0(0) > 0$ then there exists a time T^* such that $u(x,t) > 0$ for all $(x,t) \in (-1,1) \times (T^*,+\infty)$.

Proof Let us consider a Barenblatt-type solution[3] of (1.1)

$$E_{k,\rho}(x,t;x_0,t_0) = k\rho S(t)^{-1/\lambda_0}\left(1 - \left(\frac{|x-x_0|}{S(t)^{1/\lambda_0}}\right)^{p/(p-1)}\right)_+^{(p-1)/(m(p-1)-1)}, \qquad (3.1)$$

where $k > 0, \rho > 0, t_0 > 0, x_0 \in (-1,1)$ and

$$\lambda_0 = (m+1)(p-1), S(t) = \lambda_0\left(\frac{mp}{m(p-1)-1}\right)^{p-1}(k\rho)^{m(p-1)-1}(t-t_0) + \rho^{\lambda_0}, (t \geqslant t_0).$$

From $u_0(0) > 0$, it follows that $u_0(x) > k, \forall x \in (-\rho,\rho) \subset I$ for sufficiently small constant $k > 0$. Clearly, we have

$$\frac{\partial E_{k,\rho}(x,t;0,0)}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial E^m_{k,\rho}(x,t;0,0)}{\partial x}\right|^{p-2}\frac{\partial E^m_{k,\rho}(x,t;0,0)}{\partial x}\right), \qquad (3.2)$$

$$E_{k,\rho}(x,0;0,0) = k\left(1 - \left(\frac{|x|}{\rho}\right)^{p/(p-1)}\right)_+^{(p-1)/(m(p-1)-1)} \leqslant k \leqslant u_0(x), \forall x \in I, \qquad (3.3)$$

$E_{k,\rho}(\pm 1,t;0,0) = u(\pm 1,t) = 0, \forall\, t \in (0, T^*)$. (3.4)

Where

$$T^* = \frac{1-\rho^{\lambda_0}}{\lambda_0\left(\frac{mp}{m(p-1)-1}\right)^{p-1}(k\rho)^{m(p-1)-1}}.$$

By Lemma 2.1 and using (3.2)~(3.4), we get $u(x,t) \geqslant E_{k,\rho}(x,t;0,0)$ for all $(x,t) \in I \times (0,T^*)$, which implies that

$$u(x,T^*) \geqslant 0, \tag{3.5}$$

for all $x \in (-1,1)$. By (3.5) and Lemma 2.3 we get $u(x,t) \geqslant 0$ for all $(x,t) \in (-1,1) \times (T^*, +\infty)$. Thus the proof is completed.

Lemma 3.2 Let u be a weak solution of the problems (1.1)~(1.3) and $m(p-1) > 1$. If

$$u_0(x_0) > 0, x_0 \neq 0, \tag{3.6}$$

then there exists a time T such that

$$u(0,T) > 0. \tag{3.7}$$

Proof Without loss of generality, we assume that $0 < x_0 < 1$.

From (3.6), it follows that

$$u_0(x) > k_0 = \frac{1}{2}u_0(x_0) > 0,$$

for all $x \in (x_0 - \rho_0, x_0 + \rho_0)$ with some small positive number ρ_0.

Similar to (3.2)~(3.4), we have

$$\frac{\partial E_{k_0,\rho_0}(x,t;x_0,0)}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial E^m_{k_0,\rho_0}(x,t;x_0,0)}{\partial x}\right|^{p-2}\frac{\partial E^m_{k_0,\rho_0}(x,t;x_0,0)}{\partial x}\right),$$

$$E_{k_0,\rho_0}(x,0;x_0,0) = k_0\rho_0 S(0)^{-1/\lambda_0}\left(1-\left(\frac{|x-x_0|}{S(0)^{1/\lambda_0}}\right)^{p/(p-1)}\right)_+^{(p-1)/(m(p-1)-1)} = k_0\left(1-\left(\frac{|x-x_0|}{\rho_0}\right)^{p/(p-1)}\right)_+^{(p-1)/(m(p-1)-1)} \leqslant k_0 \leqslant u_0(x), x \in (2x_0-1,1),$$

$$E_{k_0,\rho_0}(\pm 1,t;x_0,0) = k_0\rho_0 S(t)^{-1/\lambda_0}\left(1-\left(\frac{|\pm 1 - x_0|}{S(t)^{1/\lambda_0}}\right)^{p/(p-1)}\right)_+^{(p-1)/(m(p-1)-1)}.$$

From

$$1-\left(\frac{|\pm 1 - x_0|}{S(t)^{1/\lambda_0}}\right)^{p/(p-1)} = 0,$$

we take

$$T_{01} = \frac{|1-x_0|^{\lambda_0} - \rho_0^{\lambda_0}}{\lambda_0\left(\frac{mp}{m(p-1)-1}\right)^{p-1}(k_0\rho_0)^{m(p-1)-1}},$$

$$T_{02} = \frac{|1+x_0|^{\lambda_0} - \rho_0^{\lambda_0}}{\lambda_0\left(\frac{mp}{m(p-1)-1}\right)^{p-1}(k_0\rho_0)^{m(p-1)-1}}, \tag{3.8}$$

$$T_0 = \min\{T_{01}, T_{02}\}.$$

then we have

$$E_{k_0,\rho_0}(\pm 1,t;x_0,0) = u(\pm 1,t) = 0, t \in (0,T_0).$$

By Lemma 2.1, we get $u(x,t) \geqslant E_{k_0,\rho_0}(x,t;x_0,0)$ for all $(x,t) \in (2x_0-1,1) \times (0,T_0)$, which implies

$$u(x,T_0) > 0, \tag{3.9}$$

for all $x \in (2x_0 - 1, 1)$.

If $x_0 < \frac{1}{2}$, then by (3.9), we have (3.7).

If $x_0 \geqslant \frac{1}{2}$, we define $x_1 = x_0 - d_k$.

Where $x_0 > d_k = \frac{1}{k}(1-x_0) > 0$ and $k > 2$ is a positive integer.

Similar to (3.9), there exists a time T_1 such that

$$u(x,T_0+T_1) > 0, \tag{3.10}$$

for all $x \in (2x_1 - 1, 1)$.

If $x_1 < \frac{1}{2}$, then by (3.10) we have (3.7).

If $x_1 \geqslant \frac{1}{2}$, we define $x_2 = x_1 - d_k = x_0 - 2d_k$.

Similar to (3.9), there exists a time T_2 such that

$$u(x,T_0+T_1+T_2) > 0, \tag{3.11}$$

for all $x \in (2x_2 - 1, 1)$.

Repeating the above process, we can find two positive integer n^* and k^* such that $x_{n^*} = x_0 - d_{k^*}$, $-1 < 2x_{n^*} - 1 < 0$ and $u(x, T_0 + T_1 + T_2 + \cdots + T_{n^*}) > 0$for all $x \in (2x_{n^*} - 1, 1)$. Thus the proof is completed.

Proof of theorem 1.1 From Lemmas 3.1 and 3.2, there exists a time T such that $u(x,T) > 0$ for all $x \in (-1,1)$. By Lemma 2.3, we have the conclusion of Theorem 1.1. Thus the proof is completed.

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