A New Lower Bound of Trivially Non-contractible Edges in a Contraction Critical 5-connected Graph*

收缩临界5连通图中平凡不可收缩边的新下界

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Abstract: It is proved that any contraction critical 5-connected graph on n vertices has at least $n+1$ trivially non-contractible edges.

Key words: graph, connected graph, contractible edge, contraction critical, fragment

摘要:证明 n 个顶点的收缩临界5连通图中至少有 $n+1$ 条平凡不可收缩边.

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1 Introduction

We only consider finite and simple graphs. Basically we follow the terminology of Reference [1]. Let $G=(V,E)$ be a graph with vertex set V and the edge set E. For a vertex $x\in V$, we denote the neighbourhood of x by $N(x)$, which is the set of vertices adjacent to x. $d(x)=|N(x)|$ denotes the degree of x. $E(x)$ denotes the set of the edges incident with x. For a nonempty set $F\subseteq V$, let $N(F)=(\bigcup_{x\in F}N(x))-F$ and $\overline{F}=V-(F\cup N(F))$. The set F or the subgraph induced by F is called a fragment of G if $\overline{F}\neq\phi$ and $|N(F)|=\kappa(G)$, where $\kappa(G)$ denotes the connectivity number of G. We also call F a $N(F)$-fragment. For the subsets S and T of V, we denote by $E(S,T)$ the set of edges between S and T. If $S=\{x\}$, then we simply write $E(x,T)$ instead of $E(\{x\},T)$. For a connected graph G, a subset $S\subseteq V(G)$ is said to be a cut-set of G, if $G-S$ is not connected. A cut-set S is called a k-cut-set if $|S|=k$.

Let G be a k-connected non-complete graph (where $k\geqslant 2$), an edge of G is called k-contractible if its contraction results still in a k-connected graph. An edge that is not k-contractible is called a non-contractible edge. If G does not have a k-contractible edge, then G is called contraction critical k-connected. It is easy to see that a k-connected graph G is contraction critical if and only if for each edge $e=xy$ of G, G has a k-cut-set containing $\{x,y\}$. If the contraction of $e\in E$ results in a graph with minimum degree $k-1$, then e is called trivially non-contractible. In other words, e is trivially non-contractible if and only if the two end vertices of e have a common neighbour of degree k.

In 1961, Tutte[2] proved that any 3-connected graph with order at least 5 had a 3-contractible edge. On the other hand, Thomassen[3] showed that for $k\geqslant 4$ there were infinitely many k-connected k-regular graphs in which there was no a k-contractible edge. So it is nature to study the structure of contraction critical k-connected graphs. The contraction critical 4-connected graphs were characterized by Martinov[4],

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which are two special classes of 4-regular graphs. For $k \geqslant 5$, the characterization for the contraction critical k-connected graphs seems to be very hard. In general, Egawa[5] showed that every contraction critical k-connected graph had a vertex of degree at most $[\frac{5k}{4}] - 1$. By Egawa's result, the minimum degree of a contraction critical 5-connected graph is 5. In this direction, more results have been obtained.

Theorem 1[6,7] Let G be a contraction critical 5-connected graph. Then each vertex of G has a neighbour of degree 5, and thus G has at least $|G|/5$ vertices of degree 5.

Su[8] proved the results that any vertex of a contraction critical 5-connected graph G had at least two neighbours of degree 5, and thus G had at least $2|G|/5$ vertices of degree 5. The number 'two' is the best possible shown in Reference [9].

Thomassen[3] proved that any contraction critical k-connected graph contained one triangle. Mader[10] obtained that every contraction critical k-connected graph G contained at least $|G|/3$ triangles. Recently, Kriesell[11] further improved Mader's result to that a contraction critical k- connected graph G contained at least $2|G|/3$ triangles.

From these results, we may expect that a contraction critical 5-connected graph has many trivially non-contractible edges. Motivated by this, Ando[12] considered the distribution of the trivially non-contractible edges in a contraction critical 5-connected graph and proved the following results.

Theorem 2[12] Any contraction critical 5-connected graph G has least $|G|/2$ trivially non-contractible edges.

Ando guessed that the lower bound of Theorem 2 can be improved to $|G|$, and even to $2|G|$, and he proposed his problem in the China-Japan Joint Conference on Discrete Geometry, Combinatorics and Graph Theory (2005). Here we prove the following result.

Theorem 3 Any contraction critical 5-connected graph G has at least $|G| + 1$ trivially non-contractible edges.

2 Proof of Theorem 3

In this section, we will prove Theorem 3. Before that we state some lemmas. For the fragments, we have the following properties.

Lemma 1[10] Let F and F' be two distinct fragments of $G, T = N(F), T' = N(F')$.

(1)If $F \cap F' \neq \phi$, then $|F \cap T'| \geqslant |\overline{F'} \cap T|$, $|F' \cap T| \geqslant |\overline{F} \cap T'|$.

(2)If $F \cap F' \neq \phi \neq \overline{F} \cap \overline{F'}$, then both $F \cap F'$ and $\overline{F} \cap \overline{F'}$ are fragments of G, and $N(F \cap F') = (T \cap T') \cup (T \cap F') \cup (F \cap T')$, $N(\overline{F} \cap \overline{F'}) = (T \cap T') \cup (T \cap \overline{F'}) \cup (\overline{F} \cap T')$.

Lemma 2[7] Let G be a contraction critical 5-connected graph and F a fragment of G. If $w \in N(F)$, $N(w) \cap N(F) \neq \phi$ and $|\overline{F}| \geqslant 2$, then $N(w) \cap (F \cup N(F))$ contains a vertex of degree 5.

Lemma 3[11] Let G be a contraction critical 5-connected graph and A a fragment of cardinality 2 in G. If $N(A)$ has two vertices $x \neq y$ such that $|N(x) \cap A| = |N(y) \cap A| = 1$, then one of x, y has a neighbour of degree 5 in $N(A) - \{x, y\}$.

In the following, we always assume that G is a contraction critical 5-connected graph. Let E^* denote the set of the trivially non-contractible edges of G, and let $\beta(x) = |E(x) \cap E^*|$. Denote $V_5 := \{v \in V(G) | d(v) = 5\}$.

Let $T = \{a_1, a_2, x, y, z\}$ be a 5-cut-set of G, $A = \{u, v\}$ a component of $G - T$ such that $\{u, v, z\} \subseteq V_5(G)$, $G[A] \cong K_2$, $N(u) = \{a_1, a_2, x, z, v\}$, $N(v) = \{a_1, a_2, z, y, u\}$, $yz \in E$, $yx \in E$, there may exist other edges among the vertices of T. We call the induced subgraph $G[V(A) \cup T]$ a K_2- configuration with centre x.

Proposition 1 Let x be a vertex of G such that $\beta(x) \leqslant 1$, or $\beta(x) = 2$ and $E(x) \cap E^*$ be in a triangle, then G has a K_2- configuration with centre x.

Proof Since $\beta(x) \leqslant 1$ or $\beta(x) = 2$ and $E(x) \cap E^*$ be in a triangle, for any fragment A with $x \in N(A)$ we have that $E(x, A) \cap E^* = \phi$ or $E(x, \overline{A}) \cap E^* = \phi$. We consider the fragments A such that $N(A)$ contains an edge of $E(x) - E^*$ and $E(x, A) \cap E^* = \phi$, among them we choose A such that $|A|$ is minimum. As $N(A)$ contains an edge of $E(x) - E^*$, so $|A| \geqslant 2$ and $|\overline{A}| \geqslant 2$.

Claim 1 $|A| = 2$.

Proof We only need to prove that $|A| \leqslant 2$. Let $u \in N(x) \cap A$, then $xu \in E(x) - E^*$. Let S be a

5-cut-set containing $\{x,u\}$, B a S-fragment of G. Since xu is an element of $E(x)-E^*$, then $|B|\geqslant 2$, $|\overline{B}|\geqslant 2$. Let $T=N(A)$.

We first assume $A\cap B\neq\phi$. If $\overline{A}\cap\overline{B}\neq\phi$, by Lemma 1(2) and $A\cap B\neq\phi$, we get $A\cap B$ is a fragment, note $A\cap B$ is a fragment such that $T':=N(A\cap B)=(A\cap S)\cup(S\cap T)\cup(B\cap T)$ contains an element of $E(x)-E^*$ and $E(x,A\cap B)\cap E^*=\phi$. Further more, $u\in A\cap S$, $|A\cap B|\leqslant|A|-|A\cap S|\leqslant|A|-1$, which contradicts the choice of fragment A, so $\overline{A}\cap\overline{B}=\phi$ and $A\cap B$ isn't a fragment, thus $|(A\cap S)\cup(S\cap T)\cup(B\cap T)|\geqslant 6$. If $A\cap\overline{B}\neq\phi$, arguing similarly, we can obtain $\overline{A}\cap B=\phi$. Thus $|\overline{A}\cap S|=|\overline{A}|\geqslant 2$. Since $|(A\cap S)\cup(S\cap T)\cup(B\cap T)|\geqslant 6$, $|S|=|(A\cap S)\cup(S\cap T)\cup(\overline{A}\cap S)|=5$, we have $|B\cap T|\geqslant|\overline{A}\cap S)|+1\geqslant 3$. $|\overline{B}\cap T|\geqslant 3$ follows similarly, note $x\in S\cap T$, thus $|T|=|B\cap T|+|S\cap T|+|\overline{B}\cap T|\geqslant 7$, a contradiction. So $A\cap\overline{B}=\phi$, and $|\overline{B}\cap T|=|\overline{B}|\geqslant 2$. Then we get $|A\cap S|\geqslant|\overline{B}\cap T)|+1\geqslant 3$ in the same way. Since $x\in S\cap T$, $|S|=5$, then we have $|\overline{A}\cap S|\leqslant 1$. But we know $|\overline{A}|\geqslant 2$, so $\overline{A}\cap B\neq\phi$. By Lemma 1(1), we get $|\overline{A}\cap S|\geqslant|T\cap\overline{B}|\geqslant 2$, which contradicts $|\overline{A}\cap S|\leqslant 1$. So $A\cap B=\phi$. By symmetry, we have $A\cap\overline{B}=\phi$.

Then we have $A\cap B=0=A\cap\overline{B}$, That's to say, $A\subseteq S$. If $|A|\geqslant 3$, then $|\overline{A}\cap S|\leqslant 1$. For $|\overline{A}|\geqslant 2$, either $\overline{A}\cap B$ or $\overline{A}\cap\overline{B}$ is nonempty. We assume that $\overline{A}\cap B\neq\phi$ without loss generality. By Lemma 1(1), we get $|B\cap T|\geqslant|A\cap S|\geqslant 3$. If $\overline{A}\cap\overline{B}\neq\phi$, by Lemma 1(1), we get $|\overline{B}\cap T|\geqslant|A\cap S|\geqslant 3$. We have $|T|=|B\cap T|+|S\cap T|+|\overline{B}\cap T|\geqslant 3+1+3=7$, a contradiction. So $\overline{A}\cap\overline{B}=\phi$, then $|\overline{B}\cap T|=|\overline{B}|\geqslant 2$, thus $|T|=|B\cap T|+|S\cap T|+|\overline{B}\cap T|\geqslant 3+1+2=6$, a contradiction. Then we have $|A|\leqslant 2$, so $|A|=2$. The proof of Claim 1 is completed.

By Claim 1, we let $A=\{u,v\}$, $T=\{a_1,a_2,x,y,z\}$, $xu,yx\in E(x)-E^*$. First, we claim $d(u)=5$. If $d(u)\neq 5$, then $d(u)=6$, thus $N(u)=\{v\}\cup T$. If $d(v)=5$, then $N(v)=\{a_1,a_2,z,y,u\}$; otherwise, $xu\in E^*$ is a trivially non-contractible edge, which contradicts our assumption $E(x,A)\cap E^*=\phi$, so $N(x)\cap A\cap V_5=\phi$. If $d(v)=6$, we also have $N(x)\cap A\cap V_5=\phi$. Then by Lemma 2, there exists a vertex $w\in\{a_1,a_2,y,z\}$ and $d(w)=5$, $xw\in E$, thus $xu\in E(x,A)\cap E^*$, a contradiction. So we have $d(u)=5$. Because $xy\notin E^*$, we have $uy\notin E$, then $N(u)=\{a_1,a_2,x,z,v\}$. We next claim $d(v)=5$. If $d(v)\neq 5$, then $d(v)=6$, thus $xv\in E^*$, contradicts $E(x,A)\cap E^*=\phi$.

From above, we know $|N(x)\cap A|=|N(y)\cap A|=1$, by Lemma 3, there exists a vertex $w\in V_5\cap\{a_1,a_2,z\}$ and $w\in N(x)$ or $w\in N(y)$. If $w\in N(x)$, then $ux\in E^*$, a contradiction, so $w\in N(y)$. Without loss of generality, we let w be z. From above, we obtain a K_2- configuration with centre x, and the proof of Proposition 1 is completed.

Proof of Theorem 3 Let $V_0=\{x\in V(G)\mid\beta(x)\leqslant 1\}$, $V_2=\{x\in V(G)\mid\beta(x)=2\}$, By Proposition 1, for every $x\in V_0$, G has a K_2- configuration with centre x. In this K_2- configuration, there exists a $u_x\in A\cap N(x)$, $N(u_x)\cap V_0=\{x\}$ and $\beta(u_x)=4$. For each $x\in V_0$, we can find only one u_x, thus we define $V_1=\{u_x\mid x\in V_0\}$. Obviously, $|V_1|=|V_0|$. Let $V_3=V(G)-(V_0\cup V_1\cap V_2)$. In addition, in this K_2-configuration, there is a $v_x\in A-N(x)$ and $\beta(v_x)=5$. So if $|V_3|=0$ then $|V_0|=0$.

Considering the cardinality of E^*, we have $2|E^*|=\sum_{v\in V}\beta(v)=\sum_{v\in V_0}\beta(v)+\sum_{v\in V_1}\beta(v)+\sum_{v\in V_2}\beta(v)+\sum_{v\in V_3}\beta(v)\geqslant 0+4|V_1|+2|V_2|+3|V_3|=2|V_0|+2|V_1|+2|V_2|+2|V_3|+|V_3|=2|V(G)|+|V_3|\geqslant 2|V(G)|$, thus $|E^*|\geqslant|G|$, the equality holds only when $V_3=V_0=\phi$ and $\beta(v)=2$ for each $v\in V$.

Next, we claim that the equality doesn't hold. If not, there is a contraction critical 5-connected graph G with $|E^*|=|G|$. We can easily to see that $|E^*|=|G|$ if and only if for each $v\in V$, $\beta(v)=2$, thus locally to see, the structure of every vertex x and $E(x)\cap E^*=\{xy,xz\}$ has only following 3 cases:

Case(1) xyz be a triangle with $y\in V_5$, $z\in V_5$;

Case(2) uyx and vzx are two edge-disjoint triangles, $z,y\notin V_5$ and $u,v\in V_5$;

Case(3) xwy and xwz are two triangles which have a common edge, $w\in V_5$, $y\notin V_5$, $z\notin V_5$.

If Case (1) occurs, by Proposition 1, G has a K_2-configuration with center x, thus there is a vertex $u\in V$, $\beta(u)=4$, a contradiction. Then Case (1) doesn't occur for every vertex x, thus for each $x \in V$, $N(x) \cap V_5$ is an independent set.

Now we only consider the vertex of degree 5, that's to say $x \in V_5$.

If Case (2) occurs, for y, $\{u,x\} \subseteq N(y) \cap V_5$, $ux \in E$, contradicts that $N(y) \cap V_5$ is an independent set. If Case (3) occurs, for y, $\{x,w\} \subseteq N(y) \cap V_5$, $xw \in E$, contradicts that $N(y) \cap V_5$ is an independent set.

So for $x \in V_5$, none of the Cases (1), (2), (3) occurs, but V_5 isn't empty, a contradiction. So our assumption is absurd, which means $|E^*| \geqslant |G| + 1$. The proof is completed.

Corollary 1 G is a contraction critical 5-connected graph of order n, then G contains at least $n/3$ triangles.

Proof If the number of triangles in G is less than $n/3$, then the number of edges in triangles is less than n. But every trivially non-contractible edge is in a triangle, by Theorem 3, G has at least n edges in triangles, this is a contradiction.

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调控干细胞分裂的新分子被发现

每一个干细胞都具有分裂成为两种细胞的能力，其中之一的细胞会在分裂后快速分化发育，成为体细胞的一部分；另一种细胞则仍然保有干细胞的能力，随时保持分裂行为中母细胞的能力，因此，干细胞才能源源不绝地在体内随时待命，适时修补破损的细胞与组织。

然而，分裂中的干细胞为了确定母细胞在分裂周期中顺利产生子细胞，就必须沿着事先出现的轴线，进行所谓的分裂动作。如果干细胞在任何一个动作中出现了异常现象，就可能会导致分裂不完全，不仅会失去干细胞原先的功用，还可能导致肿瘤细胞产生。干细胞的分裂行为，与肿瘤细胞的增生失控，是一体两面的事。

巴塞罗那生物医学研究院的科学家通过高分辨率显微镜的辅助，以数百张的影像观察细胞分裂的过程。他们除了观察到中心体在分裂的过程中占有关键性的角色之外，还看到了一个原本被定义成肿瘤抑制分子的蛋白质，参与了中心体的调控过程。他们找到了一个从未发现过的，可以主宰和影响干细胞分裂行为的调控机制。如果细胞的分裂真是如此，那么研究细胞变异，发生癌化的机制就又多了一条切入的线索。

(据科学网)