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On B-H Partial Ordering of Matrices and Their Exponents
关于矩阵及其方幂矩阵的 B-H偏序

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Abstract The relationship between the partial ordering of two matrices and the partial ordering of their exponentsand A^k and B^k ($k = 2, 3$) is discussed in the sense of B-H partial ordering

Key words matrix, partial ordering, B-H partial ordering

摘要:研究矩阵 A, B 及其方幂矩阵 A^k 和 B^k($k = 2, 3$) 的 B-H偏序之间的关系.

关键词:矩阵 偏序 B-H偏序
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1 Introduction and preliminaries

Let $C_{m\times n}$ stand for the set of $m\times n$ complex matrices, and $H_{m\times n}$ for the subset of $C_{m\times n}$ consisting of Hermitian matrices. The symbols A^H, $R(A)$ and $r(A)$ stand for the conjugate transpose, range and the rank of $A\in C_{m\times n}$, respectively.

For matrices $A, B\in C_{m\times n}$, star partial ordering, minus partial ordering and B-H ordering are defined as

$$A \overset{H}{\leqslant} B \Leftrightarrow AA^H = BA^H, A^HA = A^HB;$$
$$A \overset{-}{\leqslant} B \Leftrightarrow r(B - A) = r(B) - r(A);$$
$$A \leqslant B \Leftrightarrow A < B, AB^HA = AA^HA,$$

respectively.

For non-negative definite matrices, Baksalary and Pukelsheim[1] had discussed the relationships of partial orderings of $A \overset{L}{\leqslant} B$, $A \overset{H}{\leqslant} B$ and $A \overset{-}{\leqslant} B$ with that ones between A^2 and B^2. In the present paper we discuss the relationship of partial ordering of $A \leqslant B$ with that ones between A^k and B^k ($k = 2, 3$). Some related results for Hermitian matrices are obtained.

We first introduce a few preliminary lemmas.

Lemma 1.1[2] Let $A, B\in C_{m\times n}$ with the rank a and b respectively, then

$A \overset{H}{\leqslant} B$ if and only if $A = U\begin{pmatrix} D & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}V^H$, $B = U\begin{pmatrix} D & 0 & 0\\ 0 & E & 0\\ 0 & 0 & 0\end{pmatrix}V^H$,

where D, E are diagonal positive definite matrices, U, V are unitary matrices.

Lemma 1.2[3] Let $A, B\in C_{m\times n}$ with the rank a and b respectively, then

$A \leqslant B$ if and only if $A = U\begin{pmatrix} D & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}V^H$, $B = U\begin{pmatrix} D & DR & 0\\ SD & SDR + E & 0\\ 0 & 0 & 0\end{pmatrix}V^H$,

where D, E are diagonal positive definite matrices, U, V are unitary matrices.

2 The main results

The main results are released as follows.

Theorem 2.1 Let $A, B\in H_{m\times n}$, if

$$A = U\begin{pmatrix} D & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}V^H,$$

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$$B = U\begin{pmatrix} D & DR & 0 \\ SD & SDR + E & 0 \\ 0 & 0 & 0 \end{pmatrix} V^H, \qquad (2.1)$$

then

$$V^H U = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_5 & 0 \\ 0 & 0 & T_9 \end{pmatrix}. \qquad (2.2)$$

Where T_1, T_5, T_9 are unitary matrices, and $T_1RS = DR^HT_5^H$, $T_5RD = S^HDT_1^H$, $T_5(RDS + E) = T_5^H(S^HDR^H + E)$, $T_1DT_1 = D$. (2.3)

Proof Since $A^H = A$, then

$$V^HU\begin{pmatrix} D & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^HU = \begin{pmatrix} D & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $V^HU = \begin{pmatrix} T_1 & T_2 & T_3 \\ T_4 & T_5 & T_6 \\ T_7 & T_8 & T_9 \end{pmatrix}$, then

$$\begin{pmatrix} T_1DT_1 & T_1DT_2 & T_1DT_3 \\ T_4DT_1 & T_4DT_5 & T_4DT_6 \\ T_7DT_1 & T_7DT_8 & T_7DT_9 \end{pmatrix} = \begin{pmatrix} D & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore

$T_2 = 0, T_3 = 0, T_4 = 0, T_7, T_1DT_1 = D$.

Similarly, since $B^H = B$, we have

$$\begin{pmatrix} T_1D & T_1RS & 0 \\ T_5RD & T_5(RDS + E) & 0 \\ T_8RD & T_8(RDS + E) & 0 \end{pmatrix} = \begin{pmatrix} DT_1^H & DR^HT_5^H & DR^HT_8^H \\ S^HDT_1^H & (RDS + E)^HT_5^H & (RDS + E)^HT_8^H \\ 0 & 0 & 0 \end{pmatrix}.$$

This relation implies that $T_8 = 0$.

Since V^HU is a unitary matrix, we have $T_6 = 0$, so Formulae (2.2) and (2.3) are held.

Theorem 2.2 Let $A, B \in H_{n\times n}$, then $A \leqslant B$, $A^2 \leqslant B^2$ if and only if $A^H \leqslant B$.

Proof If $A^H \leqslant B$, obviously $A \leqslant B$, $A^2 \leqslant B^2$. If $A \leqslant B$, $A^2 \leqslant B^2$, using theorem 2.1 yields,

$$A^2 = U\begin{pmatrix} DT^1D & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^H,$$

$$B^2 = U\begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} V^H,$$

T_{11}: $DT_1D + DST_5RD$; T_{21}: $RDT_1 + RDST_5RD + ET_5RD$; T_{12}: $DT_1DS + DST_5RDS + DST_5E$; T_{22}: $RDT_1DS + (RDS + E)T_5(RDS + E)$.

Since $A^2 \leqslant B^2$, we obtain $DST_5RD = 0$. From Formula (2.3), we have $S^HS = 0$, So $S = 0$, and $R = 0$. By Lemma 1.1, we obtain $A^H \leqslant B$.

Corollary 2.3[4] Let A, B be non-negative definite matrices, then $A \leqslant B$, $A^2 \leqslant B^2$ if and only if $A^H \leqslant B$.

Corollary 2.4 Let $A, B \in H_{n\times n}$, we have

(1) If $A \leqslant B$, $A^2 \leqslant B^2$, then $AB = BA$.

(2) If $AB = BA$ and $A \leqslant B$, then $A^2 \leqslant B^2$.

Proof (1) If $A \leqslant B$, $A^2 \leqslant B^2$, then $A^H \leqslant B$, and there exists unitary matrix U. Such as

$$A = U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} U^H, B = U\begin{pmatrix} D & 0 \\ 0 & F \end{pmatrix} U^H. \qquad (2.4)$$

Where D is a diagonal positive definite matrix, and F is a diagonal matrix.

$$AB = U\begin{pmatrix} D^2 & 0 \\ 0 & 0 \end{pmatrix} U^H = BA.$$

(2) As $A \leqslant B$, then from Formulae (2.1) and (2.2), we have

$$\begin{pmatrix} DT_1D & DTD_1R & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} DT_1D & 0 & 0 \\ SDTD_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $R = 0, S = 0$, and $A^H \leqslant B$. From Formula (2.4), it is easily to prove $A^{2H} \leqslant B^2$, so $A^2 \leqslant B^2$.

We extend the results in References [5].

Remark 2.5 If and $AB = BA$ and $A^2 \leqslant B^2$, but $A \leqslant B$ may be not true.

That can be understanded easily from the following example (even for star partial ordering).

Example 2.6 $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$. obviously, $AB = BA$ and $A^2 \leqslant B^2$, but $A \leqslant B$ is not held.

Theorem 2.7 Let A, B be non-negative definite matrices, then

$A \leqslant B$, $A^3 \leqslant B^3$ if and only if $A^H \leqslant B$.

Proof If $A^H \leqslant B$, obviously, $A \leqslant B$, $A^3 \leqslant B^3$.

Conversely if $A \leqslant B$, $A^3 \leqslant B^3$, using Lemma 1.2, we have $A \leqslant B \Leftrightarrow A, B$ have the following decomposition

$$A = U\begin{pmatrix} D & 0 & 0 \\ 0 & 0 & \\ 0 & 0 & 0 \end{pmatrix} U^H,$$

$$B = U\begin{pmatrix} D & DR^H & 0 \\ RD & E + RDR^H & 0 \\ 0 & 0 & 0 \end{pmatrix} U^*. \qquad (2.5)$$

Where D, E are diagonal positive definite matrices, U is a unitary matrix.

Since $A^3 \leqslant B^3$, it follows that

$$R^HRD + DR^HR + R^HRDR^HR + R^HER = 0. \qquad (2.6)$$

Suppose $R \neq 0$, let

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$$R^H R = T\,\mathrm{diag}(\lambda_1, \cdots, \lambda_t, 0, \cdots, 0)\, T^H,$$

where $T = (t_{ij})_{n\times n}$ is a unite matrix, and $\lambda_1 > 0$.

Since $C = (c_{ij})_{n\times n} = T^H R^H RDR^H RDT$, $G = (g_{ij})_{n\times n} = T^H R^H ERT$ are non-negative definite matrices, then $c_{11} \leqslant 0$, $g_{11} \geqslant 0$.

Let $F = \mathrm{diag}(\lambda_1, \cdots, \lambda_t, 0, \cdots, 0)\, T^H DT + T^H DT\, \mathrm{diag}(\lambda_1, \cdots, \lambda_t, 0, \cdots, 0)$,

we obtain $f_{11} = 2\lambda_1 (\sum_{i=1}^{n} d_i t_{1i} t_{1i}^H)$.

From $\sum_{i=1}^{n} t_{1i} t_{1i}^H = 1, d_i > 0$, we have

$$f_{11} + c_{11} + g_{11} > 0,$$

which is a contradiction to Formula (2.6).

So $R = 0$, and $A^H \leqslant B$.

Corollary 2.8 Let A, B be non-negative definite matrices, if $A \leqslant B$, $A^3 \leqslant B^3$, then $AB = BA$.

Proof If $A \leqslant B$, $A^3 \leqslant B^3$, then $A^H \leqslant B$, so $AB = BA$.

3 Conclude

The relation between the minus partial ordering of two matrices A and B relates to the B-H partial ordering of theirs exponent A^k and B^k $(k = 2, 3)$ are given, But our method seems unavailable for the general case, and we pose an open question.

Question As a consequence of above corollary, we conjecture

$$A \leqslant B, A^k \leqslant B^k (k \geqslant 4) \Rightarrow AB = BA.$$

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