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The Existence of Almost Periodic Solutions for Nonlinear Volterra Integrodifferential Equations
非线性 Volterra积分微分方程概周期解的存在性*

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Abstract A nonlinear Volterra integrodifferential equation is investigated. The existence and uniqueness of almost periodic solutions for the equation is obtained.

Key words Volterra integrodifferential equation, almost periodic solution, existence

摘要 研究非线性 Volterra方程概周期解，得到方程存在唯一概周期解的一组充分条件.

关键词 Volterra积分微分方程 概周期解 存在性
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1 Introduction

Consider the nonlinear Volterra integrodifferential equation

$$x'(t) = A(t)x(t) + \int_{-\infty}^{t} C(t,s,x(s))\,ds + g(t, x_t) \tag{1}$$

and the perturbed equation

$$x'(t) = A(t)x(t) + \int_{-\infty}^{t} C(t,s,x(s))\,ds + g(t, x_t) + p(t), \tag{2}$$

where $-\infty < s \leqslant t, t \in R, x \in R^n$, x_t is defined by the relation $x_t = x(t+s), -\infty < s \leqslant t$. Specially, if $C(t,s,x(s)) = C(t,s)x(s)$, and $g(t,x_t) = g(t)$, Burton[1] and Huang[2] have investigated the existence of periodic solutions. In this paper, we consider the existence of almost periodic solutions for equation (1) under the condition that the solutions of the equation are bounded and eventually totally stable. The existence theorem of an almost periodic solution is obtained.

Definition 1 $C(t,s,x)$ is said to be almost periodic in t uniformly for (s,x), if for any $\varepsilon > 0$ and any compact set K in $R \times R^n$, there exists a $L = L(\varepsilon, K) > 0$ such that any interval of length L contains a τ for which

$$|C(t+\tau, s, x) - C(t,s,x)| \leqslant \varepsilon, \tag{3}$$

for all $t \in R$ and all $(s,x) \in K$.

Definition 2 Let $f(t)$ be a continuous function defined on R. $f(t)$ is said to be asymptotically almost periodic if it is a sum of a continuous almost periodic function $p(t)$ and a continuous function $q(t)$ defined on R which tends to zero as $t \to \infty$, that is

$$f(t) = p(t) + q(t). \tag{4}$$

It is well known that $f(t)$ is asymptotically almost periodic if and only if for any sequence $\{\tau_k'\}$ such that $\tau_k' \to \infty$ as $k \to \infty$, there exists a subsequence $\{\tau_k\}$ for which $f(t+\tau_k)$ converges uniformly on R.

Let B denote the vector space of bounded continuous functions mapping $(-\infty, 0]$ into R^n, and for any $\varphi, \psi \in B$, we set

$$\rho(\varphi, \psi) = \sum_{j=1}^{\infty} \rho_j(\varphi, \psi) / [2^j (1 + \rho_j(\varphi, \psi))],$$

where $\rho_j(\varphi, \psi) = \sup_{-j \leqslant s \leqslant 0} |\varphi(s) - \psi(s)|$. Clearly, $\rho(\varphi_n, \varphi) \to 0$ as $n \to \infty$ if and only if $\varphi_n(s) \to \varphi(s)$ uniformly on any compact subset of $(-\infty, 0]$ as $n \to \infty$. We denote by (B, ρ) the space of bounded continuous function $\varphi: (-\infty, 0] \to R^n$ with metric ρ.

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2 Main results

For the equations (1) and (2), we impose the following assumptions:

(i) $A(t)$ is a $n \times n$ continuous almost periodic matrix, $\lambda_i(t)(i = 1, 2, \cdots, n)$ represent the eigenvalues of $A(t)$, $\lambda_i(t) \leqslant -a(t) < 0 (i = 1, 2, \cdots, n)$ for any $t \in R$.

(ii) $C(t, s, x)$ is continuous and almost periodic in t uniformly for (s, x), $s \leqslant t$, and $(t, s, x) \in R \times R \times R^n$. Moreover, $|C(t,s,x)| \leqslant D(t,s) \sup_{s \in (-\infty, t]} |x(s)|$, where $x(s)$ is continuous on $(-\infty, t]$ for any $t \in R$ such that $|x(s)| \leqslant \_$ on this interval. $\int_{-\infty}^{t} D(t,s) ds \leqslant b(t)$. There exists a positive constant k such that $\sup_{s \in (-\infty, t]} |x(s)| \leqslant kx(t)$, and $kb(t) < a(t)$ for any $t \in R$.

(iii) $g(t, h): R \times (B, d) \to R^n$ is continuous in t and h, and for any $r > 0$ there exists a continuous function $U(t)$ such that $U(t) \to 0$ as $t \to \infty$ and $|g(t, h)| \leqslant U(t)$, whenever $|h(s)| \leqslant r$ for all $s \in (-\infty, t]$.

Under the above assumptions (i) ~ (iii), if $t_0 \in R$ and $H \in B$, there exists a solution of (1) which passes through (t_0, H). Moreover, a solution $x(t)$ can be continuable up to $t = \infty$ if it remains in a compact set in R^n, because $x'(t)$ is bounded as long as $x(t)$ remains in a compact set in R^n.

Let S be a compact set in R^n, then we have

Definition 3 The bounded solution $u(t)$ of equation (1) is said to be eventually totally (S, d)-stable, if for any $X > 0$ there exist $T = T(X) \geqslant 0$ and $W = W(X) > 0$ such that if $t_0 \geqslant T$, $d(u_{t_0}, x_{t_0}) < W$ and $p(t)$ is any continuous function which satisfies $|p(t)| < W$ on $[t_0, \infty)$, then $d(u_t, x_t) < X$ for all $t \geqslant t_0$, where $x(t)$ is a solution of equation (2) such that $x_{t_0}(s) \in S$ for all $s \leqslant t$.

Lemma 1[3] If there exists an asymptotically almost periodic solution for an almost periodic differential equation, then there exists an almost periodic solution of the equation.

Therefore, in order to show that there exists an almost periodic solution of the equation (1), we first show that there exists an asymptotically almost periodic solution for this equation.

Lemma 2 Suppose that the conditions (i) ~ (iii) are satisfied, then there exists a bounded solution of equation (1).

Proof Let $x(t)$ be a solution of equation (1), consider a Liapunov function $V(t) = \frac{1}{2}x^2(t)$, then we have

$V'_{(1)}(t) = x(t)[A(t)x(t) + \int_{-\infty}^{t} C(t,s,x(s)) ds + g(t, x_t)] \leqslant -a(t)x^2(t) + kx^2(t)\int_{-\infty}^{t} D(t,s) ds + g(t, x_t)x(t) \leqslant -(a(t) - kb(t))x^2(t) + U(t)x(t)$. (5)

Since $kb(t) < a(t)$, and $U(t) \to 0$ as $t \to \infty$, therefore, if t is sufficiently large, there exist positive constants c and M such that $V'_{(1)}(t) \leqslant -cV(t) + M$, this means $x(t)$ is bounded.

Theorem 1 Under the assumptions (i) ~ (iii), if the bounded solution $x(t)$ is eventually totally (S, d)-stable, then $x(t)$ is asymptotically almost periodic. So there exists an almost periodic solution of equation (1).

Proof We can write equation (1) as

$$x'(t) = A(t)x(t) + \int_{-\infty}^{0} C(t, s, x(t+s)) ds + g(t, x_t). \quad (6)$$

Let t_k be a sequence such that $t_k \to \infty$ as $k \to \infty$. If we set $x^k = x(t + t_k)$, $k = 1, 2, \cdots$, then $x^k(t)$ is a solution of the following equation

$$x'(t) = A(t + t_k)x(t) + \int_{-\infty}^{0} C(t + t_k, s, x(t+s)) ds + g(t + t_k, x_t). \quad (7)$$

Clearly, $x^k(t)$ remains in S. Since $x(t)$ is eventually totally (S, d)-stable, $x^k(t)$ is also eventually totally (S, d)-stable.

For any given $X > 0$, there exists a positive integer $k_1 = k_1(X)$ such that $t_k \geqslant T$ if $k \geqslant k_1$. Taking a subsequence if necessary, we can assume that $x^k(t)$ converges uniformly on any compact set in $(-\infty, 0]$ as $k \to \infty$. Therefore there exists a positive integer $k_2 = k_2(X)$ such that if $k, m \geqslant k_2$, $d(x_0^k, x_0^m) < W$. Clearly $x^m(t) = x(t + t_m)$ is a solution of

$$x'(t) = A(t + t_k)x(t) + \int_{-\infty}^{0} C(t + t_k, s, x(t+s)) ds + g(t + t_k, x_t) + p(t) \quad (8)$$

and $x^m(t) \in S$ for all $t \in R$, where

$$p(t) = A(t + t_m)x^m(t) + \int_{-\infty}^{0} C(t + t_m, s, x^m(t+s)) ds + g(t + t_m, x_t^m) - A(t + t_k)x^m(t) - \int_{-\infty}^{0} C(t +$$

$t_k, s, x^m(t+s))\mathrm{d}s - g(t+t_k, x_t^m)$. (9)

We shall show that there exists a positive integer $k_0 = k_0(X)$ such that if $k, m \geqslant k_0$, $|p(t)| < W$ for $t \geqslant 0$. Since $x(t)$ for any $t \in R$ is bounded, there exists a $V > 0$ such that $|x| \leqslant V$ for all $x \in S$. It is clear that $|x^k(t)| \leqslant V$ and $|x^m(t)| \leqslant V$ for all $t \in R$. By assumption (ii), there exists a $l = l(V, X) > 0$ such that for all $t \in R$

$$\int_{-\infty}^{-l} |C(t+t_m, s, x^m(t+s))| \mathrm{d}s \leqslant \frac{W}{5} \quad (10)$$

and

$$\int_{-\infty}^{-l} |C(t+t_k, s, x^m(t+s))| \mathrm{d}s \leqslant \frac{W}{5}. \quad (11)$$

Since $A(t)$ and $C(t,s,x)$ are almost periodic in t and $g(t, h) \to 0$ as $t \to \infty$, for this l there exists a positive integer $k_0 = k_0(X) \geqslant \max(k_1, k_2)$ such that if $k, m \geqslant k_0$,

$$|C(t+t_m, s, x^m(t+s)) - C(t+t_k, s, x^m(t+s))| \leqslant \frac{W}{5l}, (-l \leqslant s \leqslant 0), \quad (12)$$

$$|A(t+t_m)x^m(t) - A(t+t_k)x^m(t)| \leqslant \frac{W}{5}, \quad (13)$$

and for $t \geqslant 0$,

$$|g(t+t_m, x_t^m) - g(t+t_k, x_t^m)| \leqslant \frac{W}{5}. \quad (14)$$

Since we have

$$\left|\int_{-\infty}^{0} C(t+t_m, s, x^m(t+s))\mathrm{d}s - \int_{-\infty}^{0} C(t+t_k, s, x^m(t+s))\mathrm{d}s\right| \leqslant \int_{-\infty}^{-l} |C(t+t_m, s, x^m(t+s))| \mathrm{d}s + \int_{-\infty}^{-l} |C(t+t_k, s, x^m(t+s))| \mathrm{d}s + \int_{-l}^{0} |C(t+t_m, s, x^m(t+s)) - C(t+t_k, s, x^m(t+s))| \mathrm{d}s, \quad (15)$$

then we obtain $|p(t)| < W$ for $t \geqslant 0$ if $k, m \geqslant k_0$. Since $x^m(t)$ is a solution of (8) which remains in S and $x^k(t)$ is eventually totally (S, d)-stable, we have $d(x_t^k, x_t^m) < X$ for all $t \geqslant 0$ if $k, m \geqslant k_0$. This implies that if $k, m \geqslant k_0$,

$$|x(t+t_k) - x(t+t_m)| \leqslant \sup_{s \in [-1, 0]} |x(t+t_k+s) - x(t+t_m+s)| \leqslant 4X, \quad (16)$$

for all $X \leqslant \frac{1}{4}$ and all $t \geqslant 0$. Thus we see for any sequence $\{t_k'\}$ such that $t_k' \to \infty$ as $k \to \infty$, there exists a subsequence $\{t_k\}$ of $\{t_k'\}$ for which $x(t+t_k)$ converges uniformly on $[0, \infty)$ as $k \to \infty$. This shows that $x(t)$ is asymptotically almost periodic in t. Therefore, there exists an almost periodic solution of equation (1).

In order to consider the uniqueness of almost periodic solutions of equation (1), we replace the condition (ii) by the following.

(ii)′ $C(t,s,x)$ is continuous and almost periodic in t uniformly for (s,x), $s \leqslant t$, and $(t,s,x) \in R \times R \times R^n$. Moreover, $C(t,s,x)$ satisfied Lipschitz condition, namely, $|C(t,s,x(s)) - C(t,s,y(s))| \leqslant D(t,s) \sup_{s \in (-\infty, t]} |x(s) - y(s)|$, where $x(s)$ and $y(s)$ are continuous on $(-\infty, t]$ for any $t \in R$ such that $|x(s) - y(s)| \leqslant \_$ on this interval. $\int_{-\infty}^{t} D(t,s)\mathrm{d}s \leqslant b(t)$. There exists a positive constant k such that $\sup_{s \in (-\infty, t]} |x(s) - y(s)| \leqslant k|x(t) - y(t)|$, and $kb(t) < a(t)$ for any $t \in R$.

Theorem 2 Under the assumptions (i), (ii)′, and (iii), if the bounded solution $x(t)$ of the equation (1) is eventually totally (S, d)-stable, then there exists one and only one almost periodic solution of equation (1).

Proof From (ii)′ one can deduce that condition (ii) is satisfied. Therefore, from Theorem 1, there exists one asymptotically almost periodic solution of equation (1). So there exists an almost periodic solution of the equation. Suppose that $x(t)$ and $y(t)$ are two asymptotically almost periodic solutions of equation (1), consider function $W(t) = \frac{1}{2}[x(t) - y(t)]^2$, then we have

$$W'_{(1)}(t) = (x(t) - y(t))[A(t)(x(t) - y(t)) + \int_{-\infty}^{t} (C(t,s,x(s)) - C(t,s,y(s)))\mathrm{d}s + g(t,x_t) - g(t,y_t)] \leqslant -a(t)[x(t) - y(t)]^2 + k(x(t) - y(t))^2 \int_{-\infty}^{t} D(t,s)\mathrm{d}s + (g(t,x_t) - g(t,y_t))(x(t) - y(t)) \leqslant -(a(t) - kb(t))(x(t) - y(t))^2 + 2U(t)(x(t) - y(t)). \quad (17)$$

Since $kb(t) < a(t)$ for any $t \in R$ and $U(t) \to 0$ as $t \to \infty$, therefore, we have $\lim_{t \to \infty} x(t) = y(t)$. From references [4], $x(t)$ and $y(t)$ are two asymptotically almost periodic solutions of equation (1), then $x(t) - y(t)$ is still an asymptotically almost periodic solutions of equation (1). Now suppose that $x(t)$ is a sum of a continuous almost periodic function $p(t)$ and a continuous function $q(t)$ defined on R which tends to zero as $t \to \infty$, and $y(t)$ is a sum of a continuous almost periodic function $r(t)$ and a continuous function $s(t)$ defined on R which tends to zero as $t \to \infty$, then

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需的临界耦合强度.此结论可以推广到多个节点的耦合网络的同步中.相信具有 N 个节点的耦合网络中每一个节点与其它节点的耦合方式是通过节点间状态方程中一个变量进行耦合的,那么整个网络达到同步时所需的耦合强度是远远大于节点间通过所有变量进行耦合的网络.究其原因,可能是随着网络的不断增大,尽管网络中每一个节点都要受到与它相耦合的其它节点的束缚,但是如果网络中的节点间的耦合方式不同,那么节点间受到的束缚程度也是不同的.正如上面所讨论的,如果网络中的节点与节点之间是通过单变量进行耦合的,那么它们在空间中的自由度增大,因此可能通过较大的耦合强度来使整个网络达到同步.而对于网络节点间是通过多变量耦合,随着耦合变量的增多,使网络节点在空间中的自由度减小,此时每个节点受到与它耦合节点的约束也就越强,因此,多变量耦合系统所需的临界耦合强度就越小.

同样,也可以将上述分析应用于另外一些与之不同的耦合形式中.譬如,以 2个节点的网络为例, $X_{1,2}$ 是联系 2个节点的耦合参数,通过它可以设置不同方向的耦合形式,等等.本文中只是针对单变量及多变量耦合的双向耦合 2种情况进行研究和对比,另外还有许多形式更加复杂、意义更为深广的情形还有待于我们进一步研究.

参考文献

1 Pecorra L M, Carroll T L. Synchronization in Chaotic Systems. Phys Rev Lett, 1990, 64: 821-824.

2 Pecorra L M, Carroll T L. Driving System with Chaotic Signals. Phys Rew A, 1991, 44: 2374-2383.

3 Ott E, Grebogic, Yorke J A. Control Chaos. Phys Rew Lett, 1990, 64: 1196-1199.

4 Watts D J. Small Worlds: The Dynamics of Networks between Order and Randomness. Princeton: Princeton University Press, 1999.

5 Lago-Fernandez L F, Huerta R, Corbacho F, et al. A Fast response and temporal coherent oscillations in small-world networks. Phys Rev Lett, 2000, 84(12): 2758-2761.

6 Wang X F, Chen G. Synchronization in small-world dynamical networks. Int J Bifurcation and Chaos, 2002, 12: 187-192.

7 Gade P M, Hu C-K. Synchronous chaos in coupled map with small world interactions. Phys Rev E, 2000, 62(5): 6409-6413.

8 Wang X F, Chen G. Synchronization in scale-free dynamical networks: robustness and fragility. IEEE Trans Circuits Syst I, 2002, 49: 54-62.

9 胡 岗,萧井华,郑志刚编著.混沌控制.上海:上海科技教育出版社,2000.

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we have

$$\lim_{t\to\infty}(x(t)-y(t))=\lim_{t\to\infty}[(p(t)+q(t))-(r(t)+s(t))]=\lim_{t\to\infty}[p(t)-r(t)]+\lim_{t\to\infty}[q(t)-s(t)]=\lim_{t\to\infty}[p(t)-r(t)]. \quad (18)$$

Since $\lim_{t\to\infty}x(t)=y(t)$, therefore, $\lim_{t\to\infty}p(t)=r(t)$. The proof is complete.

Example Consider the scalar equation

$$x'(t)=-(4+\cos t-\cos\pi t)x(t)+\frac{1}{4}\int_{-\infty}^{t}(\cos t+\cos\pi t)e^{-t+s}\frac{x(s)}{1+x^2(s)}ds+\frac{x_t e^{-t^2}}{1+e^{x_t^2}}, \quad (19)$$

where $x_t=x(t+s)$, $-\infty<s\leqslant t$. One can verify that the conditions of Theorem 2 are satisfied. So there exists one and only one almost periodic solution of equation (19).

References

1 Burton T A. Unified boundedness, periodicity and stability in ordinary and functional differential equations. Annali Mat Pur Appl, 1986, CXLV: 129-158.

2 Huang Qichang. On the existence of periodic solutions for functional differential equations with infinite delays. Science of China, 1984, 10A: 882-889.

3 Zheng Zuxiu. Theory of Functional Differential Equations. Hefei: Anhui Educational Press, 1994.

4 He Congyou. Almost Periodic Differential Equations. Beijing: High Educational Press, 1992.

5 Hamaya Y. Total stability property in limiting equations of integrodifferential equations. Funkcial Ekvac, 1990, 33: 345-362.

6 Burton T A. Periodic solutions of integrodifferential equations. J London Math Soc, 1985, 31: 527-548.

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