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A Weak *n*-Engel Condition and the *p*-nilpotency of Finite Groups*
有限群的弱 *n*-Engle条件和 *p*-幂零性

Zhong Xianggui
钟祥贵

(Dept. of Math., Guangxi Normal Univ., 3 Yucailu, Guilin, Guangxi, 541004, China)
(广西师范大学数学系 桂林市育才路 3号 541004)

Abstract Let G be a finite group. For a weak n-Engel condition of G, we mean that $[x,_n y]\in Z(G)$ for two elements x and y of G, where n is a positive integer. A subgroup H of G is called c-supplemented in G if there exists a subgroup K of G such that $HK=G$ and $H\cap K\leqslant Core_G(H)$. In this paper, we discuss the sufficient conditions of a p-nilpotent group by using supplementation of the cyclic subgroups of order 4 under some assumptions weaker than the n-Engel condition on minimal subgroups of G.

Key words finite group, Engel condition, p-nilpotent group, c-supplemented subgroup

摘要 有限群 G的一个弱 n-Engle条件是指: 对于 G的 2个元素 x,y和某个非负整数 n, $[x,_n y]\in Z(G)$成立,如果存在 G的一个子群 K满足 $HK=G$和 $H\cap K\leqslant Core_G(H)$,则 G的一个子群 H称为 c-可补的.利用极小子群的弱 n-Engle条件和 4阶循环子群的 c-可补性,讨论了 G的 p-幂零性.

关键词 有限群 Engle条件 p-幂零群 c-可补子群
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1 Introduction

For many questions in group theory, especially for group structure problems, it is helpful if one knows some criteria for p-nilpotency of a group. It shows that for an odd p, a group G is p-nilpotent provided that all elements of G of order p lie in the centre of G and that G is 2-nilpotent provided that all elements of G of order 2 or 4 lie in the centre of G[1]. In recent years, the p-nilpotent structures of a group G (namely, for a Sylow p-subgroup P of G, there exists a normal subgroup N of G such that $G=PN$ and $P\cap N=1$) have been widely studied[2~6].

For convenience, we introduce some basic terms and definitions. A minimal subgroup of a group G is a subgroup of prime order, and a p-element of G is an element in G of prime p-power order. Let $[x,y]$ be commutator for x and y. We denote $[x,_0 y]$ the element x and $[x,_n y]$ the commutator $[x,_{(n-1)}y,y]$, where n is a positive integer. We call a subgroup H c-supplemented in G, if there exists a subgroup N of G such that $HN=G$ and $H\cap N\leqslant Core_G(H)$. In this paper, we discuss the sufficient conditions of a p-nilpotent group by using supplementation of the cyclic subgroups of order 4 under some assumptions weaker than the n-Engel condition on minimal subgroups.

Throughout, all groups are finite.

2 Preliminaries

Lemma 1[6] Let H be c-supplemented in G. Then

(1) If $H\leqslant K\leqslant G$, then H is c-supplemented in K;

(2) If $K\trianglelefteq G$ and $K\leqslant H$, then H/K is c-supplemented in G/K.

Lemma 2 Let G be a minimal non-2-nilpotent

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group (non-2-nilpotent group all of whose proper subgroups are 2-nilpotent), and if a cyclic subgroup $< x >$ of G of order 4 is c-supplemented in G. Then $< x >$ is a normal subgroup of G.

Proof By c-supplementation condition, there is a subgroup L of G such that $G = < x > L$ with $< x > \cap L \leqslant Core_G(< x >)$. If $< x > \cap L$ is of order 4, then $< x > \leqslant L$, and so $< x > = < x > \cap L = Core_G(< x >) \trianglelefteq G$; if $| < x > \cap L| = 2$, then $|G/L| = 2$ and L is a non-trivial proper normal subgroup of G. Hence, $< x >$ is a normal subgroup of G by Reference [7, Lemma 4]; if $< x > \cap L = 1$, it is easy to prove that $|G\ LZ| = 2$ where $Z = < x^2 >$ and $LZ \trianglelefteq G$. Notice that L is a proper subgroup of G so L is nilpotent. Let L_q be a normal 2-complement of L, then L_q is also a normal 2-complement of LZ and so $L_q \trianglelefteq G$. In this case, L_q is a Sylow q-subgroup of G and $G = P \times Q$, a final contradiction, and the lemma is proved.

Lemma 3[8] Let G be a minimal non-p-nilpotent group. Then

(1) $G = PQ$, where P is a normal Sylow p-subgroup for a prime p and Q is a cyclic Sylow q-subgroup for a prime $q (q \neq p)$;

(2) If P is abelian, then P is an elementary abelian p-group;

(3) If $p > 2$, then the exponent of P is p. If $p = 2$, then the exponent of P is 2 or 4;

(4) $c \in P - \Phi(P)$ if and only if $[c, b] \neq 1$, where b is an element of Q which generates Q;

(5) $Z(G) = \Phi(G) = \Phi(P) \times \Phi(Q)$.

Lemma 4 Let $G = AB$, where A is a nilpotent normal subgroup of G and B is a nilpotent subgroup of G with $(|A|, |B|) = 1$. If there is a positive integer n such that $[x, _n y] \in Z(G)$, where x is a p-element of A and y is a q-element of B, then $[x, y] = 1$.

Proof Since $[x, y] \in Z(G)$ means $[x, _2 y] \in Z(G)$, we may assume $n \geqslant 2$. Let $a = [x, _{n-2} y]$, by the hypothesis we have $[(y^{-1})^a y, y] = [a, y, y] \in Z(G)$, It follows that $(y^{-1})^a y y = y (y^{-1})^a y z$ for some $z \in Z(G)$. That is, $(y^{-1})^a y = y (y^{-1})^a z$, i. e., $[(y^{-1})^a, y] \in Z(G)$. We consider the following cases:

(1) If $|B|_2 = 1$ or $|B|_2 > 1$ and q is an odd prime, then $o(y) | C^2_{o(y)}$ and $((y^{-1})^a y)^{o(y)} = ((y^{-1})^a)^{o(y)} y^{o(y)} ([y, (y^{-1})^a]^{o(y)})^{C^2_{o(y)}/o(y)} = 1$.

Hence $(y^{-1})^a y$ is a q-element of G.

(2) If $|B|_2 > 1$ and q is 2, then $((y^{-1})^a y)^{2o(y)} = 1$ and $(y^{-1})^a y$ is a 2-element of G.

On the other hand, since A is a normal subgroup of G, $(y^{-1})^a y = [a, y] \in A$. And $(y^{-1})^a y$ is a p-element. Therefore, by the above discussions, $[a, y] = (y^{-1})^a y = 1$. That is, $[x, _{n-1} y] = 1$. Now, a simple induction on n, we have $[x, y] = 1$.

3 Main results

Theorem 1 Let G be a finite group. If every cyclic subgroup of G of order 4 is c-supplemented in G and if for every element x of order p in G and any q-element $(q \neq p)\, y$ in G, there is a positive integer n such that $[x, _n y]$ either is a p-element or lies in $Z(G)$, then G is p-nilpotent.

Proof Suppose that the result is false and let G be a counterexample of minimal order. Because every cyclic subgroup of order 4 of each proper subgroup of G is c-supplemented in this subgroup; on the other hand, since every element x of order p or every $(q \neq p)$ q-element y of each proper subgroup K of G is an element of order p or a q-element of G, then we have $[x, _n y]$ either is a p-element or lies in $Z(G) \cap K \leqslant Z(K)$. Thus, every proper subgroup of G is p-nilpotent and G is a minimal non-p-nilpotent group, $G = PQ$ with properties given in Lemma 3.

For every element c of order p of P and an element b of Q, which generates Q by the assumption, there is a positive integer n such that $[c, _n b]$ is a p'-element or lies in $Z(G)$. If $[c, _n b]$ is a p'-element, we have $[c, _n b] = 1$ since P is a normal subgroup of G. Now Lemma 4 yields that $[c, b] = 1$, so $c \in \Phi(P) \leqslant Z(G)$ from Lemma 3. On the other hand, Lemma 2 implies that $< x > Q$ is a proper subgroup of G and so $< x > Q = < x > \times Q$ for every element x of order 4 of P. Thus, $P \leqslant C_G(Q)$ from Lemma 3(3). In other words, $G = P \times Q$, which contradicts G is non-p-nilpotent. This contradiction shows that the theorem is proved.

Theorem 2 Let G be a finite group and N a normal subgroup of G such that G/N is p-nilpotent. If every cyclic subgroup of N of order 4 is c-supplemented in G and if for every element x of N of order p and q-element $(q \neq p)\, y$ of G, there is a positive integer n such that $[x, _n y]$ either is a p-element or lies in $Z(G)$. Then G is p-nilpotent.

Proof Suppose the theorem is false and let G be a counterexample with minimal order. For each proper subgroup K of G, because every cyclic subgroup of order 4 of N is c-supplemented in G, then, using Lemma 1, we know that every cyclic subgroup of order 4 of $K\cap N$ is c-supplemented in this subgroup K. On the other hand, because each subgroup of a p-nilpotent group is p-nilpotent, then from $K/K\cap N\cong KN/N$, we have that $K/K\cap N$ is a p-nilpotent subgroup of G/N. It is easy to see that K and $K\cap N$ satisfy the assumption of Theorem 2 and so K is a p-nilpotent subgroup of G. It follows that $G=PQ$, where P and Q have properties in Lemma 3.

Now, if $N=1$, then, by the given condition, G is p-nilpotent, a contradiction. If $N=G$, then from the given condition and Theorem 1, we obtain that G is also p-nilpotent, again a contradiction. Thus, N is a proper subgroup G and so N is nilpotent. Let $N=N_p\times N_q$, where N_p is a Sylow p-subgroup of N and N_q is a Sylow q-subgroup of N. We claim that N_p is a proper subgroup of P and N_q is a proper subgroup of Q. In fact, since N is a normal subgroup of G. We have that N_p and N_q are normal subgroups of G, it follows that $N_p\leqslant P$ and $N_q\leqslant Q$. Hence, we only need to prove $N_p<P$. Suppose $N_p=P$, then $P\leqslant N$. If P is abelian, then P is an elementary Abel group from Lemma 3 (2). It follows by the given condition and Lemma 4 that $P\leqslant C_G(Q)$, contrary to that G is a counterexample. Hence, P is nonabelian and the exponent of P is 2 or 4. Let x be an element of P of order 2, then by the given condition and Lemma 4 we obtain $x\in C_G(Q)$. If x is an element of P of order 4, then Lemma 2 gives that $<x>Q$ is a proper subgroup of G, and so $<x>Q$ is nilpotent. In particular, $<x>\leqslant C_G(Q)$. So, all elements of P are contained in $C_G(Q)$. Thus, G is p-nilpotent. This contradiction shows that P is a proper subgroup of G.

By the above discussions, it is easy to see that N_pQ is a proper subgroup of G, and so $N_pQ=N_p\times Q$. This means that $N_p\leqslant H(P)\leqslant Z(G)$ from Lemma 3(4). On the other hand, obviously, $PN_q=P\times N_q$ and N_q is contained in $C_G(P)$. Hence, $N\leqslant Z(G)$ and $G/Z(G)$ is p-nilpotent since G/N is p-nilpotent. It follows that $G/Z(G)$, and hence G is nilpotent. This is a final contradiction and the proof of the theorem is complete.

Theorem 2 may be considered as a generalization of Theorem 3 in Reference [5], Theorem 2 in Reference [9] and Theorem 5 in Reference[7].

Theorem 3 Let G be a finite group and P an arbitrary p-subgroup of G. If $p=2$, then every cyclic subgroup of P of order 4 is c-supplemented in $N_G(P)$, and if for every element x of order p of P and q-element $(q\neq p)$ y of $N_G(P)$, there is a positive integer n such that $[x,_n y]$ either is a p'-element or lies in $Z(G)$. Then G is p-nilpotent.

Proof Let y be an arbitrary q-element $(q\neq p)$ of $N_G(P)$. We can consider the group $P<y>$. Since $P<y>\leqslant N_G(P)$, if $p=2$, then every cyclic subgroup of $P<y>$ of order 4 is a cyclic subgroup of P of order 4, and hence, it is c-supplemented in $P<y>$ from Lemma 1. Additionally, for every element x of order p of $P<y>$ we have $x\in P$. Therefore, let y' be any q-element of $P<y>$, then, since for some positive integer n, $[x,_n y']$ either is a p'-element and $[x,_n y']=1$ or $[x,_n y']\in Z(N_G(P))\cap P<y>$, which implies that $P<y>$ is p-nilpotent from Theorem 1. So that $P<y>$ is a nilpotent subgroup of G. It follows that $N_G(P)/C_G(P)$ is a p-group. By the Frobenius' Theorem, G is p-nilpotent. The proof is completed.

It is easy to see that a normal subgroup of G is a c-supplemented subgroup of G and G is nilpotent if and only if G is p-nilpotent for every prime $p\,|\,|G|$. As a direct application of the above Theorems, we will obtain a series of known results on nilpotency and p-nilpotency of a group. For instance, we have the following corollary.

Corollary Let G be a finite group and P a p-subgroup of G. If for every element x of order p or 4 (if $p=2$) of P and q-element $(q\neq p)$ y of $N_G(P)$, there is a positive integer n such that $[x,_n y]=1$. Then $N_G(P)/C_G(P)$ is p-group[8].

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$$B'_2 = E g(X_1) h(X_1,\theta_0) + 2\sum_{j=1}^{\infty} E g(X_1) h(X_{1+j},\theta_0).$$

证毕 .

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