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Parseval-type Equality of Dyadic Wavelet and Its Extension
二进小波的 Parseval型等式及其推广

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Abstract A pair of Parseval-type equalities of quasi-dyadic wavelets and a Parseval-type equality of dyadic wavelet are presented.

Key words Parseval-type equality, dyadic wavelet, quasi-dyadic wavelet

摘要 给出一对拟二进小波的 Parseval型等式和一个二进小波的 Parseval型等式 .

关键词 Parseval型等式 二进小波 拟二进小波
中图法分类号 O173.1

1 Introduction

Denote the Fourier transform of f by $\hat{f}$. And $\|\circ\|$ and $(\circ, \circ)$ are the norm and the inner product in $L^2(R)$, respectively.

Let $J \in L^2(R)$. If there are two positive numbers A and B such that

$$A \leqslant \sum_m |\hat{J}(\frac{k}{2^m})|^2 \leqslant B, \qquad \text{a. e.} \tag{1}$$

(here and later, $\sum_m = \sum_{m=-\infty}^{\infty}$), then J is called a dyadic wavelet[1,2]. The inequality (1) is called the stability condition.

In this paper, we shall extend the above definition of dyadic wavelet as follows:

Definition 1 Let $J_1, J_2 \in L^2(R)$. If there are two positive numbers A and B such that

$$A \leqslant \left|\sum_m \overline{\hat{J}_1(\frac{k}{2^m})}\hat{J}_2(\frac{k}{2^m})\right| \leqslant \sum_m \left|\overline{\hat{J}_1(\frac{k}{2^m})}\hat{J}_2(\frac{k}{2^m})\right| \leqslant B, \qquad \text{a. e.} \tag{2}$$

then $\{J_1(t), J_2(t)\}$ is said to be a pair of quasi-dyadic wavelets. $J_1(t)$ and $J_2(t)$ are called quasi-dyadic wavelets.

If $J(t)$ is a dyadic wavelet, the pair $\{J(t), J(t)\}$ is a pair of quasi-dyadic wavelets. Thus, a dyadic wavelet $J(t)$ is certainly a quasi-dyadic wavelet.

Definition 2 Let $\{J_1(t), J_2(t)\}$ be a pair of quasi-dyadic wavelets. $\{J_1^*(t), J_2^*(t)\}$ is defined by the following formulae

$$(J_1^*)^{\wedge}(k) = \frac{\hat{J}_2(k)}{\sum_m \overline{\hat{J}_1(\frac{k}{2^m})}\hat{J}_2(\frac{k}{2^m})},$$

$$(J_2^*)^{\wedge}(k) = \frac{\hat{J}_1(k)}{\overline{\sum_m \hat{J}_1(\frac{k}{2^m})\overline{\hat{J}_2(\frac{k}{2^m})}}}, \tag{3}$$

that are said to be a dual pair of $\{J_1(t), J_2(t)\}$.

Definition 3 If $J \in L^2(R)$, the transform

$$(W_J^m f)(b) = 2^m \int_{-\infty}^{\infty} f(t)\overline{J(2^m(t-b))}\,dt, b \in R \text{ for } f \in L^2(R) \tag{4}$$

that is said to the extended dyadic wavelet transform.

When $J(t)$ satisfies Inequality (1), the extended dyadic wavelet transform is just the ordinary dyadic wavelet transform[1].

2 Main results

Theorem 1 Let $J_1, J_2 \in L^1(R) \cap L^2(R)$ and $\{J_1(t), J_2(t)\}$ be a pair of quasi-dyadic wavelets and $\{J_1^*(t), J_2^*(t)\}$ be the dual pair of $\{J_1(t), J_2(t)\}$. Then for any $f \in L^2(R)$, the following pair

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of Parseval-type equalities holds:

$$\sum_m (W^m_{\psi_1} f, W^m_{\psi_1^*} f) = 2\pi \| f \|^2, \qquad (5)$$

$$\sum_m (W^m_{\psi_2} f, W^m_{\psi_2^*} f) = 2\pi \| f \|^2, \qquad (6)$$

where $W^m_\psi f$ is stated in Equality (4).

Theorem 2 Under the assumptions of Theorem 1, for any $f, g \in L^2(R)$, the following pair of generalied Parseval-type equalities holds:

$$\sum_m (W^m_{\psi_1} f, W^m_{\psi_2^*} g) = 2\pi (f, g), \qquad (7)$$

$$\sum_m (W^m_{\psi_2} f, W^m_{\psi_2^*} g) = 2\pi (f, g). \qquad (8)$$

From Theorem 2, we can get the following theorem immediately.

Theorem 3 Under the assumptions of Theorem 1, if the functions $\psi_1(t) = \psi_2(t) \overset{\text{def}}{=} \psi(t)$, then $\psi_1^*(t) = \psi_2^*(t) \overset{\text{def}}{=} \psi^*(t)$ and for any $f, g \in L^2(R)$, we have

$$\sum_m (W^m_\psi f, W^m_{\psi^*} g) = 2\pi (f, g). \qquad (9)$$

Specially, when $f = g$, we have

$$\sum_m (W^m_\psi f, W^m_{\psi^*} f) = 2\pi \| f \|^2. \qquad (10)$$

The above equalities (9) and (10) are Parseval-type equalities of dyadic wavelet. We have not found them in the published literatures by now.

3 Proof of theorems

Proof of Theorem 1 Since the proofs of Equalities (5) and (6) are similar, we only prove Equality (5) here.

By $\psi_1, \psi_2 \in L^1(R) \cap L^2(R)$, we know that their Fourier transforms satisfy

$$\hat{\psi}_1, \hat{\psi}_2 \in L^2(R) \cap L^\infty(R). \qquad (11)$$

By Formulae (2) and (3), we get $|(\psi_1^*)^\wedge(\xi)| \leqslant \frac{1}{A} |\hat{\psi}_2(\xi)|$ a.e. From this and Formula (11), we have

$$(\psi_1^*)^\wedge \in L^2(R) \cap L^\infty(R). \qquad (12)$$

Further $\psi_1^* \in L^2(R)$. Now by Equality (4) and Parseval equality of Fourier transform[3], we get

$$(W^m_{\psi_1} f)(b) = \int_{-\infty}^{\infty} \overline{\overline{\hat{f}(\xi)}\, \hat{\psi}_1(\frac{\xi}{2^m}) e^{-ib\xi}}\, d\xi. \qquad (13)$$

$$(W^m_{\psi_1^*} f)(b) = \int_{-\infty}^{\infty} \overline{\overline{\hat{f}(\xi)}\, (\psi_1^*)^\wedge(\frac{\xi}{2^m}) e^{-ib\xi}}\, d\xi. \qquad (14)$$

Set $L(\xi) = \overline{\hat{f}(\xi)}\, \hat{\psi}_1(\frac{\xi}{2^m})$ and $L^*(\xi) = \overline{\hat{f}(\xi)}\, (\psi_1^*)^\wedge(\frac{\xi}{2^m})$. By Formulae (11) and (12), we see that

$$L, L^* \in L^1(R) \cap L^2(R).$$

So by the definition of L^1 - Fourier transform, Equalities (13) and (14) are rewritten in the form

$$(W^m_{\psi_1} f)(b) = \sqrt{2\pi}\, \overline{\hat{L}(b)},$$

$$(W^m_{\psi_1^*} f)(b) = \sqrt{2\pi}\, \overline{\hat{L}^*(b)}.$$

Again by Parseval equality of Fourier transform,

$$(W^m_{\psi_1} f, W^m_{\psi_1^*} f) = 2\pi \int_{-\infty}^{\infty} \overline{\hat{L}(b)}\, \hat{L}^*(b)\, db$$

$$= 2\pi \int_{-\infty}^{\infty} \overline{L(\xi)} L^*(\xi)\, d\xi$$

$$= 2\pi \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, \overline{\hat{\psi}_1(\frac{\xi}{2^m})}\, (\psi_1^*)^\wedge(\frac{\xi}{2^m})\, d\xi.$$

By Formula (3), we see that

$$(\psi_1^*)^\wedge(\frac{\xi}{2^m}) = \frac{\hat{\psi}_2(\frac{\xi}{2^m})}{\sum_k \overline{\hat{\psi}_1(\frac{\xi}{2^{k+m}})}\, \hat{\psi}_2(\frac{\xi}{2^{k+m}})}$$

$$= \frac{\hat{\psi}_2(\frac{\xi}{2^m})}{\sum_k \overline{\hat{\psi}_1(\frac{\xi}{2^k})}\, \hat{\psi}_2(\frac{\xi}{2^k})},$$

and so by Formula (2),

$$\sum_m \left| \overline{\hat{\psi}_1(\frac{\xi}{2^m})}\, (\psi_1^*)^\wedge(\frac{\xi}{2^m}) \right| = \frac{\sum_m \left| \overline{\hat{\psi}_1(\frac{\xi}{2^m})}\, \hat{\psi}_2(\frac{\xi}{2^m}) \right|}{\left| \sum_k \overline{\hat{\psi}_1(\frac{\xi}{2^k})}\, \hat{\psi}_2(\frac{\xi}{2^k}) \right|}$$

$$\leqslant \frac{B}{A}, \text{ a.e.}$$

Again using Parseval equality of Fourier transform, we know that

$$\sum_m \int_{-\infty}^{\infty} \left| |\hat{f}(\xi)|^2\, \overline{\hat{\psi}_1(\frac{\xi}{2^m})}\, (\psi_1^*)^\wedge(\frac{\xi}{2^m}) \right| d\xi \leqslant$$

$$\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \sum_m \left| \overline{\hat{\psi}_1(\frac{\xi}{2^m})}\, (\psi_1^*)^\wedge(\frac{\xi}{2^m}) \right| d\xi \leqslant \frac{B}{A} \| \hat{f} \|^2$$

$$= \frac{B}{A} \| f \|^2.$$

So using Lebesgue's theorem of dominated convergence, we get

$$\sum_m (W^m_{\psi_1} f, W^m_{\psi_1^*} f) =$$

$$2\pi \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \sum_m \left(\overline{\hat{\psi}_1(\frac{\xi}{2^m})}\, (\psi_1^*)^\wedge(\frac{\xi}{2^m}) \right) d\xi. \qquad (15)$$

Again by Formula (3), it is easy to obtain that $\sum_m \left(\overline{\hat{\psi}_1(\frac{\xi}{2^m})}\, (\psi_1^*)^\wedge(\frac{\xi}{2^m}) \right) = 1$. From this and

Equality (15), using Parseval equality of Fourier transform, we get

$$\sum_m (W^m_{\psi_1} f, W^m_{\psi_1^*} f) = 2\pi \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk = 2\pi \| f \|^2.$$

This completes the proof of Theorem 1.

Proof of Theorem 2 Imitating the proof of Theorem 1, we can get

$$(W^m_{\psi_1} f, W^m_{\psi_1^*} g) = 2\pi \int_{-\infty}^{\infty} \hat{f}(k)\, \overline{\hat{g}(k)\hat{\psi}_1(\frac{k}{2^m})} (\psi_1^*)\hat{}\,(\frac{k}{2^m})\, dk. \qquad (16)$$

It is clear that

$$\sum_m \int_{-\infty}^{\infty} \left| \hat{f}(k)\, \overline{\hat{g}(k)\hat{\psi}_1(\frac{k}{2^m})} (\psi_1^*)\hat{}\,(\frac{k}{2^m}) \right| dk = \int_{-\infty}^{\infty} |\hat{f}(k)\, \overline{\hat{g}(k)}| \sum_m \left| \overline{\hat{\psi}_1(\frac{k}{2^m})} (\psi_1^*)\hat{}\,(\frac{k}{2^m}) \right| dk. \qquad (17)$$

In view of $\sum_m \left| \overline{\hat{\psi}_1(\frac{k}{2^m})} (\psi_1^*)\hat{}\,(\frac{k}{2^m}) \right| \leqslant \frac{B}{A}$ a. e., using Cauchy inequality, we obtain that

$$\int_{-\infty}^{\infty} |\hat{f}(k)\, \overline{\hat{g}(k)}| \sum_m \left| \overline{\hat{\psi}_1(\frac{k}{2^m})} (\psi_1^*)\hat{}\,(\frac{k}{2^m}) \right| dk$$

$$\leqslant \frac{B}{A} \int_{-\infty}^{\infty} |\hat{f}(k)\, \overline{\hat{g}(k)}|\, dk$$

$$\leqslant \frac{B}{A} \left(\int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |\hat{g}(k)|^2 dk \right)^{\frac{1}{2}} \qquad (18)$$

Again using Parseval equality of Fourier transform, we get

$$\left(\int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |\hat{g}(k)|^2 dk \right)^{\frac{1}{2}} = \| \hat{f} \| \, \| \hat{g} \| = \| f \| \, \| g \| .$$

Combining with Formulae (17) and (18), we see

$$\sum_m \int_{-\infty}^{\infty} \left| \hat{f}(k)\, \overline{\hat{g}(k)\hat{\psi}_1(\frac{k}{2^m})} (\psi_1^*)\hat{}\,(\frac{k}{2^m}) \right| dk < \infty .$$

So using Lebesgue´s theorem of dominated convergence and Equality (16), we have

$$\sum_m (W^m_{\psi_1} f, W^m_{\psi_1^*} g) = 2\pi \int_{-\infty}^{\infty} \hat{f}(k)\, \overline{\hat{g}(k)} \sum_m \left(\overline{\hat{\psi}_1(\frac{k}{2^m})} (\psi_1^*)\hat{}\,(\frac{k}{2^m}) \right) dk.$$

Again by Formula (3),

$$\sum_m \left(\overline{\hat{\psi}_1(\frac{k}{2^m})} (\psi_1^*)\hat{}\,(\frac{k}{2^m}) \right) = 1.$$

So we get Equality (7) . Similarly we can also get Equality (8) . The proof of Theorem 2 is completed.

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利用转基因鼠研究治疗黑色素瘤

荷兰和美国科学家们目前在研究与黑色素瘤治疗有关的基因研究方面取得进展,培育出了易患黑色素瘤的转基因鼠,这将有助于人们寻求治疗这种恶性皮肤癌的方法。

黑色素瘤是人类常见的皮肤恶性肿瘤之一,一般多发生在成年男性身上。黑色素瘤的发生机理目前并不十分清楚,多数人认为与遗传、长期高强度日光照射等有关。

大量研究发现,在大约一半的人类癌症中,患者染色体上都会在同一个区域发生异变。

该区域内还同时存在 2 个分别被称为 P16和 P19的基因,它们控制着蛋白质的产生,并能够抑制细胞分裂、防止癌变发生。对于这 2 个基因抗癌作用的强弱还存在一些争议,有人认为 P16基因可能不太重要。荷兰和美国的 2 个科研小组分别在有关黑色素瘤的研究报告中说,他们通过转基因手段使实验鼠体内的 P16基因不发生作用,但 P19基因保持正常。结果发现,这些转基因鼠对致癌化学物质的伤害更加敏感、更易患黑色素瘤,与单独去掉 P19基因后的情况类似。

这表明,在用转基因实验鼠治疗黑色素瘤的过程中, P16基因确实具有抑制癌症的作用。

荷兰科学家还对一批实验鼠的 P19基因进行改造,使它产生一个导致人类遗传性黑色素瘤的变异。这些易患黑色素瘤的转基因鼠可以作为模型动物,帮助科学家寻找治疗黑色素瘤的方法,并研究香烟、紫外线等致癌因素的作用。 (据科学时报)