DOI: 10.13656/j.cnki.gxkx.2002.01.002
广西科学 Guangxi Sciences 2002, 9 (1): 6- 8

广义非保守生-灭 Q过程*

An Extended Non-Conservative Birth-Death Q-Process

吴群英**

Wu Qunying

(桂林工学院基础部 桂林市建干路 12号 541004)

(Dept. of Basis Sci., Guilin Institute of Technology, 12 Jianganlu, Guilin, Guangxi, 541004, China)

摘要 给出具有突变率的广义非保守生-灭过程零流出、零流入的充分必要条件.

关键词 广义非保守生-灭过程 零流出 零流入 Q过程

中图法分类号 O211.62

Abstract The necessary and sufficient conditions of zero-exit and zero-entrance for the extended non-conservative birth-death process are obtained.

Key words extended non-conservative birth-death process, zero-exit, zero-entrance, Q-process.

设状态空间 $E = \{0, 1, 2, \cdots\}$, 考虑具有如下形式

$$Q = \begin{pmatrix} a_{11} & \lambda_0 & & & \\ \mu_1 & a_{22} & \lambda_1 & & \\ & \mu_2 & a_{33} & \lambda_2 & \\ & & \mu_3 & a_{44} & \lambda_3 \\ & & & \ddots & \ddots & \ddots \\ & & & \cdots & & \ddots \end{pmatrix} \qquad (1)$$

的非保守 Q-矩阵,其中 $\mu_0 \geqslant 0, \mu_i > 0, i \geqslant 1, \lambda_i > 0, i \geqslant 0, a_{11} = -(\lambda_0 + \mu_0 + d_0), a_{22} = -(\lambda_1 + \mu_1 + d_1), a_{33} = -(\lambda_2 + \mu_2 + d_2), a_{44} = -(\lambda_3 + \mu_3 + d_3), \cdots$, 称 $d_i \geqslant 0$为突变率, 特别当 $d_i \equiv 0$时, Q即为通常的生-灭过程,有关生-灭过程已获得许多深刻的理想结果,其详细讨论可参阅文献[1~3];由于 Q是全稳定的, Q过程一定存在, 例如 Feller最小 Q过程,参见文献[4~6]. 在讨论 Q过程的性质中,零流出、零流入的概念占有非常重要的地位,本文给出(1)式的 Q-矩阵零流出、零流入的充分必要条件.

1 主要结果的叙述

定义 称 Q-矩阵零流出, 如方程

$$\begin{cases} Qx = \lambda x, \\ 0 \leqslant x \leqslant 1. \end{cases} \qquad (2)$$

对某 $\lambda > 0$(等价于对所有 $\lambda > 0$) 只有零解.

称 Q-矩阵零流入,如方程

$$\begin{cases} yQ = \lambda y, \\ y \geqslant 0, \sum_{i=0}^{\infty} y_i < \infty. \end{cases} \qquad (3)$$

对某 $\lambda > 0$(等价于对所有 $\lambda > 0$) 只有零解.

下面简称具有(1)式的 Q-矩阵为 Q-矩阵.

定理 1 Q-矩阵零流出的充分必要条件是

$$\bar{R} \triangleq \sum_{n=1}^{\infty} \left(\frac{1 + d_n}{\lambda_n} + \frac{\mu_n}{\lambda_n} \frac{1 + d_{n-1}}{\lambda_{n-1}} + \cdots + \frac{\mu_n \mu_{n-1} \cdots \mu_2 (1 + d_1)}{\lambda_n \lambda_{n-1} \cdots \lambda_1} \right) = \infty. \qquad (4)$$

定理 2 Q-矩阵零流入的充分必要条件是

$$\bar{S} \triangleq \sum_{k=0}^{\infty} A_k = \infty. \qquad (5)$$

其中 $A_k = \sum_{i=0}^{k} \frac{F_k^{(i)}}{\mu_{i+1}}, F_k^{(k)} = 1, F_k^{(i)} = \sum_{j=i}^{k-1} \frac{q_k^{(j+1)} F_j^{(i)}}{\mu_{k+1}},$

$$0 \leqslant i < k, \qquad (6)$$

$q_n^{(k)} = d_k, k = 0, 1, \cdots, n-1, q_n^{(n)} = d_n + \lambda_n$.

注 特别,对通常的生-灭过程,即 $d_i \equiv 0$,则定理 1变为, Q-矩阵零流出的充分必要条件是

$$R = \bar{R} \triangleq \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} + \frac{\mu_n}{\lambda_n} \frac{1}{\lambda_{n-1}} + \cdots + \frac{\mu_n \mu_{n-1} \cdots \mu_2}{\lambda_n \lambda_{n-1} \cdots \lambda_1} \right) = \infty.$$

且这时有 $q_n^{(k)} = 0, k = 0, 1, \cdots, n-1, q_n^{(n)} = \lambda_n$,

2001-01-31收稿.
* 广西自然科学基金资助项目 (桂科青 9912008).
** 中南大学在读博士.

$$F_k^{(k)} = 1, F_k^{(i)} = \frac{\lambda_k}{\mu_{k+1}} F_{k-1}^{(i)} = \frac{\lambda_k\lambda_{k-1}}{\mu_{k+1}\mu_k} F_{k-2}^{(i)} = \cdots = \frac{\lambda_k\lambda_{k-1}\cdots\lambda_{i+1}}{\mu_{k+1}\mu_k\cdots\mu_{i+2}} F_i^{(i)}, 0 \leqslant i < k,$$

$$A_k = \sum_{i=0}^{k} \frac{F_k^{(i)}}{\mu_{i+1}} = \sum_{i=0}^{k} \frac{\lambda_k\lambda_{k-1}\cdots\lambda_{i+1}}{\mu_{k+1}\mu_k\cdots\mu_{i+1}};$$

定理 2变为, Q-矩阵零流入的充分必要条件是

$$\bar{S} = S = \sum_{k=0}^{\infty} \frac{1}{\mu_{k+1}} \left(1 + \frac{\lambda_k}{\mu_k} + \cdots + \frac{\lambda_k\lambda_{k-1}\cdots\lambda_2}{\mu_k\mu_{k-1}\cdots\mu_2} + \frac{\lambda_k\lambda_{k-1}\cdots\lambda_1}{\mu_{k+1}\mu_k\cdots\mu_2}\right) = \infty.$$

这些都是周知的结果[3~6],所以通常的生-灭过程有关零流出、零流入的充分必要条件是定理 1, 2的特殊情况.

2 主要结果的证明

定理 1的证明 由(1)式,方程 $Qx = \lambda x$ 即为

$$(\lambda + \lambda_0 + \mu_0 + d_0) x_0 = \lambda_0 x_1, \tag{7}$$

$$\lambda_n (x_{n+1} - x_n) = (\lambda + d_n) x_n + \mu_n (x_n - x_{n-1}), n \geqslant 1. \tag{8}$$

任取 $x_0 > 0$,因 $\lambda_0 > 0$,所以 $x_1, x_2, \cdots$,被(7)式及(8)式唯一确定,且有 $x_1 = \frac{\lambda + \lambda_0 + \mu_0 + d_0}{\lambda_0} x_0 > x_0$.

把(8)式改写成:

$$x_{n+1} - x_n = \frac{\lambda + d_n}{\lambda_n} x_n + \frac{\mu_n}{\lambda_n} (x_n - x_{n-1}), n \geqslant 1, \tag{9}$$

因为 $0 < x_0 < x_1$,由上式显然 x_n 单调不减.

记 $f_n = \frac{\lambda + d_n}{\lambda_n}, g_n = \frac{\mu_n}{\lambda_n}, a_n = f_n x_n$,

$F_n \triangleq f_n + g_n f_{n-1} + \cdots + g_n g_{n-1} \cdots g_2 f_1 + g_n g_{n-1} \cdots g_1$,

由(9)式有:

$$x_{n+1} - x_n = a_n + g_n (x_n - x_{n-1}) = a_n + g_n (a_{n-1} + g_{n-1}(x_{n-1} - x_{n-2})) = \cdots = a_n + g_n a_{n-1} + g_n g_{n-1} a_{n-2} + \cdots + g_n g_{n-1} \cdots g_1 (x_1 - x_0) = f_n x_n + g_n f_{n-1} x_{n-1} + g_n g_{n-1} f_{n-2} x_{n-2} + \cdots + g_n g_{n-1} \cdots g_1 (x_1 - x_0),$$

由 x_n 单调不减得:

$$F_n (x_1 - x_0) \leqslant x_{n+1} - x_n \leqslant F_n x_n,$$

所以 $(x_1 - x_0) \sum_{k=1}^{n} F_k \leqslant x_{n+1} - x_1, \frac{x_{n+1}}{x_n} - 1 \leqslant F_n$, (10)

又因为 $x_1 - x_0 > 0$,由此得如 $\{x_n\}$ 有界则 $\sum_{k=1}^{\infty} F_k < \infty$.

反之,如 $\sum_{k=1}^{\infty} F_k < \infty$,则由(10)式得

$$\sum_{n=1}^{\infty} \left(\frac{x_{n+1}}{x_n} - 1\right) \leqslant \sum_{n=1}^{\infty} F_n < \infty.$$

又因为 $\frac{x_{n+1}}{x_n} - 1$ 与 $\ln \frac{x_{n+1}}{x_n}$ 是等价无穷小量,所以 $\sum_{n=1}^{\infty} \ln \frac{x_{n+1}}{x_n} < \infty$. 因此,它的部分和 $S_n = \ln x_{n+1} - \ln x_1$ 有界,等价于 $\{x_n\}$ 有界.

综上所述得: $\{x_n\}$ 有界等价于:

$$\sum_{n=1}^{\infty} F_n = \sum_{n=1}^{\infty} \left(\frac{\lambda + d_n}{\lambda_n} + \frac{\mu_n}{\lambda_n} \frac{\lambda + d_{n-1}}{\lambda_{n-1}} + \cdots + \frac{\mu_n \mu_{n-1} \cdots \mu_2}{\lambda_n \lambda_{n-1} \cdots \lambda_2} \frac{\lambda + d_1}{\lambda_1}\right) < \infty,$$

等价于

$$R \triangleq \sum_{n=1}^{\infty} \left(\frac{1 + d_n}{\lambda_n} + \frac{\mu_n}{\lambda_n} \frac{1 + d_{n-1}}{\lambda_{n-1}} + \cdots + \frac{\mu_n \mu_{n-1} \cdots \mu_2 (1 + d_1)}{\lambda_n \lambda_{n-1} \cdots \lambda_1}\right) < \infty.$$

故 $\{x_n\}$ 无界,即 Q 零流出的充分必要条件是 $\bar{R} = \infty$. 定理 1证毕.

定理 2的证明

由(1)方程 $yQ = \lambda y, \lambda > 0$ 即为

$$-(\lambda_0 + \mu_0 + d_0) y_0 + \mu_1 y_1 = \lambda y_0, \tag{11}$$

$$\lambda_{n-1} y_{n-1} - (\lambda_n + \mu_n + d_n) y_n + \mu_{n+1} y_{n+1} = \lambda y_n, n \geqslant 1. \tag{12}$$

任取 $y_0 > 0$,由上方程唯一确定 $y_1, y_2, \cdots, y_n, \cdots$, Q 零流入等价于上方程无非负可和的平凡解. 在方程(12)从 1加到 n 得

$$\mu_{n+1} y_{n+1} - \mu_1 y_1 = \lambda_n y_n - \lambda_0 y_0 + \lambda \sum_{k=1}^{n} y_k + \sum_{k=1}^{n} d_k y_k,$$

结合方程(11)得

$$\mu_{n+1} y_{n+1} = \lambda_n y_n + \lambda \sum_{k=0}^{n} y_k + \sum_{k=0}^{n} d_k y_k + \mu_0 y_0,$$

令 $e_n = \sum_{k=0}^{n} y_k, e_{-1} \triangleq 0$,则 $\{e_n\}$ 单调不减,且

$$\lambda e_n + \sum_{k=0}^{n} d_k (e_k - e_{k-1}) + \lambda_n (e_n - e_{n-1}) = \mu_{n+1} (e_{n+1} - e_n) - \mu_0 e_0,$$

即

$$e_{n+1} - e_n = \left(\lambda_n (e_n - e_{n-1}) + \lambda e_n + \sum_{k=0}^{n-1} d_{k+1} (e_{k+1} - e_k) + (\mu_0 + d_0) e_0\right) / \mu_{n+1} = \left(\sum_{k=0}^{n-1} q_n^{(k+1)} (e_{k+1} - e_k) + \lambda e_n + (\mu_0 + d_0) e_0\right) / \mu_{n+1}; \tag{13}$$

为后面叙述方便,下先证明 2个引理.

引理 1 设 A_k 由(6)式定义,则

$$\frac{\sum_{k=0}^{n-1} q_n^{(k+1)} A_k + 1}{\mu_{n+1}} = A_n, n \geqslant 1.$$

证明 $\dfrac{\sum_{k=0}^{n-1} q_n^{(k+1)} A_k + 1}{\mu_{n+1}} = \sum_{k=0}^{n-1} \dfrac{q_n^{(k+1)}}{\mu_{n+1}} \sum_{i=0}^{k} \dfrac{F_k^{(i)}}{\mu_{i+1}} +$

$\dfrac{1}{\mu_{n+1}} = \sum_{i=0}^{n-1} \dfrac{1}{\mu_{i+1}} \sum_{k=i}^{n-1} \dfrac{q_n^{(k+1)} F_k^{(i)}}{\mu_{n+1}} + \dfrac{1}{\mu_{n+1}} = \sum_{i=0}^{n-1} \dfrac{1}{\mu_{i+1}} F_n^{(i)} +$

$\dfrac{F_n^{(n)}}{\mu_{n+1}} = \sum_{i=0}^{n} \dfrac{F_n^{(i)}}{\mu_{i+1}} = A_n.$

引理 **2** 设 $\{e_n: n \geqslant 1\}$ 由 (13) 式定义，则

$$\lambda e_0 A_k \leqslant e_{k+1} - e_k \leqslant (e_1 - e_0) F_k^{(0)} + (\lambda + \mu_0 + d_0) e_k A_k, k \geqslant 1. \quad (14)$$

证明 归纳法 首先证明左边的不等式，当 $k = 0$ 时，由 (11) 式及 $A_0 = 1/\mu_1, y_0 = e_0$ 得

$e_1 - e_0 = y_1 = \dfrac{\lambda + \mu_0 + \lambda_0 + d_0}{\mu_1} y_0 \geqslant \dfrac{\lambda}{\mu_1} y_0 = \lambda e_0 A_0,$

设当 $k < n$ 时已成立，由 $\{e_n\}$ 单调不减及引理 1 得

$e_{n+1} - e_n =$

$\dfrac{\sum_{k=0}^{n-1} q_n^{(k+1)} (e_{k+1} - e_k) + \lambda e_n + (\mu_0 + d_0) e_0}{\mu_{n+1}} \geqslant$

$\dfrac{\sum_{k=0}^{n-1} q_n^{(k+1)} \lambda e_0 A_k + \lambda e_n}{\mu_{n+1}} \geqslant \dfrac{\lambda e_0 \left[\sum_{k=0}^{n-1} q_n^{(k+1)} A_k + 1\right]}{\mu_{n+1}}$

$= \lambda e_0 A_n.$

所以 (14) 式左边不等式对任意 $n \geqslant 0$ 成立 .

下证 (14) 式右边不等式，当 $k = 0$ 时，由 $F_0^{(0)} = 1$ 显然有

$$e_1 - e_0 \leqslant (e_1 - e_0) F_0^{(0)} + (\lambda + \mu_0 + d_0) e_0 A_0,$$

设 $k < n$ 时已成立，则由 $\{e_n\}$ 单调不减及引理 1 得

$e_{n+1} - e_n = \left[\sum_{k=0}^{n-1} q_n^{(k+1)} (e_{k+1} - e_k) + \lambda e_n + (\mu_0 + d_0) e_0\right] / \mu_{n+1} \leqslant \left[\sum_{k=0}^{n-1} q_n^{(k+1)} ((e_1 - e_0) F_k^{(0)} + (\lambda + \mu_0 + d_0) e_k A_k) + \lambda e_n + (\mu_0 + d_0) e_0\right] / \mu_{n+1} \leqslant (e_1 - e_0) \sum_{k=0}^{n-1} \dfrac{q_n^{(k+1)} F_k^{(0)}}{\mu_{n+1}} +$

$\dfrac{(\lambda + \mu_0 + d_0) e_n \left[\sum_{k=0}^{n-1} q_n^{(k+1)} A_k + 1\right]}{\mu_{n+1}} = (e_1 - e_0) F_n^{(0)}$

$+ (\lambda + \mu_0 + d_0) e_n A_n.$

故 (14) 式右边不等式对任意 $n \geqslant 0$ 也成立，引理 2 证毕 .

继续证明定理 **2** 由引理 2 得 (14) 式成立，如 $\{e_n\}$ 有界，则由 (14) 式左边不等式得

$\lambda e_0 \sum_{k=0}^{n} A_n \leqslant e_{n+1} - e_0$，故由 $\lambda e_0 > 0$ 得

$\sum_{k=0}^{\infty} A_k < \infty.$

反之如 $\sum_{k=0}^{\infty} A_k < \infty$，因为 $A_n \geqslant \dfrac{F_n^{(0)}}{\mu_1}$ 所以 $F_n^{(0)} \leqslant \mu_1 A_n$，由 (14) 右边不等式得

$$e_{n+1} - e_n \leqslant ((e_1 - e_0)\mu_1 + (\lambda + \mu_0 + d_0) e_n) A_n,$$

由此得

$\dfrac{e_{n+1}}{e_n} - 1 \leqslant \left(\dfrac{(e_1 - e_0)\mu_1}{e_n} + \lambda + \mu_0 + d_0\right) A_n \leqslant$

$\left(\dfrac{(e_1 - e_0)\mu_1}{e_0} + \lambda + \mu_0 + d_0\right) A_n.$

因此 $\sum_{n=0}^{\infty} \left(\dfrac{e_{n+1}}{e_n} - 1\right) < \infty$，等价于 $\sum_{n=0}^{\infty} \ln \dfrac{e_{n+1}}{e_n} < \infty$，等价于 $\{\ln e_n\}$ 有界，等价于 $\{e_n\}$ 有界，即 Q 非零流入 . 故 Q 零流入的充分必要条件是 $\sum_{k=0}^{\infty} A_k = \infty$. 定理 2 证毕 .

参考文献

1 Anderson W J. Continuous-Time Markov Chains. Springer, Series in Statistics. New York: Springer-Verlag, 1991.

2 Chen M F. From Markov Chains to Non-Equilibrium Particle Systems. Singapore: World Scientific, 1994.

3 侯振挺，张汉君，邹捷中等 . 生灭过程，长沙：湖南科学技术出版社，2000.

4 Feller W. On the integro-differential equations of purely discontinuous markov Processes. Trans Ann Math Soc, 1940, 48: 488~515.

5 Feller W. On boundaries and lateral conditions for the kolmogorov differential equations. Ann Math, 1957, 65 (3): 527~570.

6 Reuter G E H. Denumerable Markov processes and the associated contraction semigroups on L. Acta Math, 1957, 97: 1~46.

(责任编辑：黎贞崇)