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A Theorem on Homeomorphic Classification of $G.M.$ $_{2n \text{ vertices}, n\geqslant 3}$

图式流形 $_{2n\text{个顶点}, n\geqslant 3}$的一个同胚分类定理

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Abstract A simple method for computing the homeomorphism class of $G.M.$ $_{2n \text{ vertices}, n\geqslant 3}$ is introduced.

Key words negative edge, negative-edge-isolated-vertex, graphlike manifold

摘要 介绍计算缩影为 $_{2n\text{个顶点}, n\geqslant 3}$的图式流形的同胚类的一个简单方法.

关键词 负边 负边弧立点 图式流形
中图法分类号 O 157.5; O 189

In $G.M.$ $_{2n \text{ vertices}, n\geqslant 3}$ when the number of negative edges is few, it can homeomorphically increase by twisting operation, while the number is great, such as more than n, some negative edges must be incident with a vertex (as every negative edge is incident with two vertices, there are $2n$ vertices only), twist the vertex, which results in the decrease of negative edges number. The purpose of this paper is to find smaller positive integer $r(0 < r \leqslant n)$, which makes the homeomorphism class of the graphlike manifold discussed equal to that of r negative edges.

Lemma When the number of negative edges is less than n, the number of negative edges can increase homeomorphically by one through the method of twisting operation (except that the number of negative edge is zero).

Proof If a vertex is incident with only one negative edge, twisted, hence the number of negative edges increases by one.

Let all the vertices be incident with two or three negative edges, they must be of the shape of negative-edge-closed-path (one or several closed paths, they are separated from one another, or two of them have a common negative edge, or two of them are linked with broken line segment of negative edges). In this case, the number of negative edges is not less than the number of the vertices which are incident with negative edges. The incident number of the vertices of p negative edges is s, $s \leqslant p$, and $p < n$, $n \geqslant 3$, so the number of negative-edge-isolated-vertices (the vertex without incident with any negative edge) $v = 2n - s \geqslant 2n - p > 2n - n = n \geqslant 3$. If two of the negative-edge-isolated-vertices at random are not adjacent, that means each negative-edge-isolated-vertex correlates with more than three positive edges. And $3v + p \geqslant 6n - 2p > 2n$. This contradiction reveals that there must be some two negative-edge-isolated-vertices being adjacent. The vertex is closely neighboring with the two negative-edge-isolated-vertices and is twisted in the closed path of negative edges, and the two negative-edge-isolated-vertices are to be twisted continuously, then, the number of negative edges increases by only one.

That is the end of proof of the lemma.

Theorem The homeomorphism class of G.M. $_{2n \text{ vertices}, n\geqslant 3}$ is equal to that of $[\frac{3}{4}n]$ negative edges, when $n = 3$, the homeomorphism class of zero negative edge is added.

Proof Take three negative edges at random, such that: [i] they can not decrease the number of negative edges through twisting their end points; [ii], they "are close to one another" —— if increasing one negative edge among them at random, then they can not be satisfied with [i]. Their relative positions will be proved to be five cases (Fig. 1):

In fact, we study negative edges a, b and c, which are satisfied with [i] and [ii], and classify all edges of the $G.M.$ by inner edges, outer edges and radio-edges. The relative positions of a, b and c are six cases

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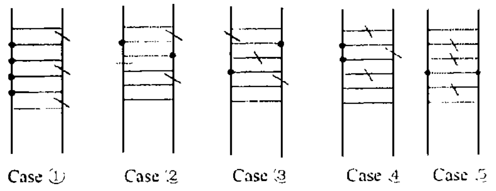


Fig. 1

The edges with bar mean negative edge. The black points express negative-edge-isolated-vertex.

as follows

(1) a,b and c are all inner (or outer) edges: there is only case ① ;

(2) a,b and c are either inner edges or outer edges: there are two cases, as showed in Fig. 2 "$-\ a\ -\ b\ -$" denotes the closed path of inner (or outer) edges, which contains a and b. In ②, the straight projection of c on "$-\ a-\ b-$" is between a and b; and in ②′, it isn't. And in ②′, twist A and B, ②′ becomes ②;

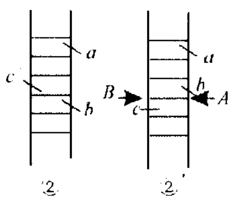


Fig. 2

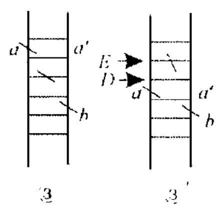


Fig. 3

(3) a,b and c is inner edge, outer edge and radioedge respectively: there are two cases, as showed in Fig. 3 a' is the projection of a. In ③, the projection of c on "$-\ a'\ -\ b\ -$" is between a' and b; and in ③′, it isn't. And in ③′, twist D and E, ③′ becomes ③;

(4) c is inner edge or outer edge, a and b are radioedges: there are two cases as showed in Fig. 4. In ④, c between the projections of a and b is on "$-\ c-$"; and in ④′, it isn't. And in ④′, twist F and G, ④′ becomes ④;

(5) a,b and c are all radioedges: there is only case ⑤;

(6) Both a and b are inner deges (or outer edges), but c is radioedge: there are two cases, as showed in Fig. 5. In ⑥′, the projection of c on "$-\ a-\ b-$" is between a and b; and in ⑥, it isn't. And in ⑥′, twist H ad I, ⑥′ becomes ⑥; in ⑥, twist M and N, ⑥ becomes ⑤.

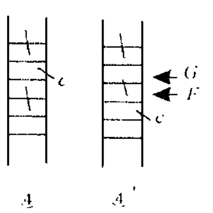


Fig. 4

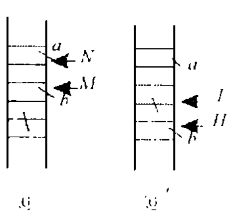


Fig. 5

So it has been proved that their real relative positions are five cases.

If projects a,b,c and those black points to inner (or outer) edges closed path, then the relative positions of their projections are at least two projections of black points among projections of a,b and c. Consequently, of r negative edges, there are not three negative edges which are not satisfied with [i], then at least there are $\frac{2}{3}r$ projections of negative-edge-isolated-vertices among projections of the r negative edges. We have

$$2r+\ \frac{2}{3}r\leqslant\ 2n,$$

so that

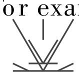
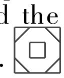
$$r\leqslant\ \frac{3}{4}n.$$

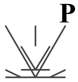
Through the proving course above, it can be easy to see that the number of negative edges must decrease homeomorphically when it is more that $[\frac{3}{4}n]$.

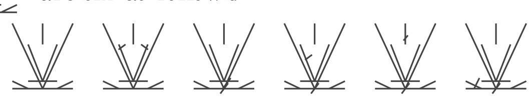
When $n\geqslant\ 3$, we have $0<[\frac{3}{4}n]<n$, from the lemma, if the number of negative edges is less $[\frac{3}{4}n]$, it could increase to $[\frac{3}{4}n]$ homeomorphically (when $n=\ 3$, except for zero negative edge, as zero negative edge does not increase to $[\frac{3}{4}\times\ 3]=\ 2$ negative edges through twisting).

Thus, homeomorphism class of the $G.M.$ discussed is equal to that of $[\frac{3}{4}n]$ negative edges, when $n=\ 3$, the homeomorphism class of zero negative edge is added.

That is the end of proof of the theorem.

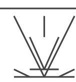
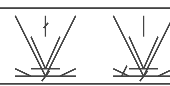
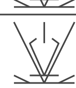
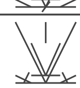
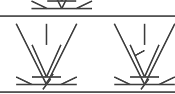
For examples, find the homeomorphism class of $G.M.$  and $G.M.$ .

Proposition The homeomorphism class of $G.M.$  are six as follows:

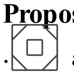


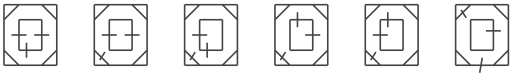
Proof According to theorem, finding homeomorphism class is equal to homeomorphism class which has 0 and 2 negative edges. Let their number of negative edge of outer, radiative and inner edges be a, b and c respectively and $a,b,c\in\ \{0,1,2\},a+\ b+\ c=\ 2$. Showed in Table 1.

Table 1

a b c	Deputy class	a b c	Deputy class
000		110	
200		020	
101			

Due to "020" ≈ "200", and the others are not homeomorphic with each other, the proposition has been proved.

Proposition The homeomorphism class of G. M.  are six as follows:

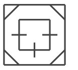
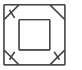
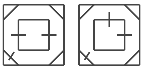
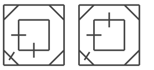
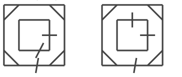
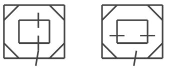
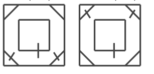
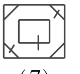
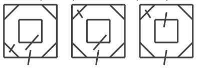
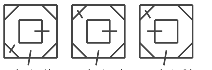


Proof By theorem, finding homeomorphism class is equal to homeomorphism class which has three negative edges. Let negative edge be outer, radiative and inner, and their numbers are respectively a, b and c, and $a, b, c \in \{0, 1, 2, 3\}$, $a + b + c = 3$. Showed in Table 2.

Through twisting, it can be easy to see that: (6) ≈ (1), (7) ≈ (1), (8) ≈ (1), (9) ≈ (2), (11) ≈ (10), (12) ≈ (1), (13) ≈ (1), (16) ≈ (14), (18) ≈ (14), (19) ≈ (15), and due to symmetry, (10) ≈ (2), (15) ≈ (4), (14) ≈ (1). So their homeomorphism classes are exactly six: (1), (2), (3), (4), (5) and (17).

The proposition has been proved.

Table 2

a b c	Deputy class	a b c	Deputy class
003	 (1)	030	 (9)
012	 (2) (4)  (3) (5)	102	 (10) (12)  (11) (13)
021	 (6) (8)  (7)	111	 (14) (16) (18)  (15) (17) (19)

Reference

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From (9), we have $|f'(x)| = \left|\int_{-1}^{x} f''(t)\,dt\right| \leqslant 2\|f''\|$, $x \in (-1, 1)$.

To sum up we get from (13) the conclusion of Lemma 1.

Let $f(x) \in C^2[-1, 1]$ and $P(x)$ is a polynomial such that $P''(x)$ is the best approximation polynomial of degree n of $f''(x)$. From $H''_{2n+1}(P; x) = P''(x)$[3], we have

$H''_{2n+1}(f; x) - f''(x) = H''_{2n+1}(f - P; x) + P''(x) - f''(x)$,

further applying Lemma 1 we immediately obtain the following Theorem 1.

Theorem 1 If $f(x) \in C^2[-1, 1]$, then for $x \in (-1, 1)$ we have

$$H''_{2n+1}(f; x) - f''(x) = \frac{O(n^2)}{1 - x^2} E_n(f''),$$

where $E_n(f'')$ is the best approximation of f'' by polynomials of degree n.

Theorem 2 If $f(x) \in C^p[-1, 1]$ and $\omega(f^{(p)}; \delta)$ is the modulus of continuity of $f^{(p)}$, then for $x \in (-1, 1)$, we have

$$H''_{2n+1}(f; x) - f''(x) = O\left(\frac{1}{n^{p-4}}\right) \frac{1}{1 - x^2} \omega\left(f^{(p)}; \frac{1}{n}\right), \quad p \geqslant 4.$$

Proof Using Theorem 1 and the Jackson theorem:

$$E_n(f'') = O\left(\frac{1}{n^{p-2}}\right) \omega\left(f^{(p)}; \frac{1}{n}\right),$$

we obtain immediately the conclusion of Theorem 2.

Similarly, by $\bar{H}_{2n+1}(f; x)$ denote Hermite-Fejér interpolation polynomials based on the zeros of $\omega(x) = (1 + x) \dfrac{\cos\frac{2n+1}{2}\theta}{\cos\frac{\theta}{2}}$, $x = \cos\theta$ (the other mixed Jacobi nodes) we also can conclude the following theorem.

If $f(x) \in C^p[-1, 1]$, then for $x \in (-1, 1)$ we have

$$\bar{H}''_{2n+1}(f; x) - f''(x) = \frac{O(n^2)}{1 - x^2} E_n(f'') = O\left(\frac{1}{n^{p-4}}\right) \frac{1}{1 - x^2} \omega\left(f^{(p)}; \frac{1}{n}\right), \quad p \geqslant 4.$$

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