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On the Class Length of Elements of Prime Power Order in Finite Groups*
关于群的素数幂阶元的共轭类长

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Abstract Let G be a finite group. Using the classification of the finite simple groups we obtain information on the structure of G under some assumptions on the lengths of conjugacy classes of elements of G of prime power order.

Key words finite group, element, conjugacy class length

摘要 讨论素数幂阶元的共轭类长对群结构的影响，改进了一系列已知结果，定理的证明依赖有限单群分类.

关键词 有限群 元素 共轭类长
中图法分类号 O 152.1

In this note G denotes always a finite group. Let $\mathrm{Con}(G)$ be the set of all the conjugacy classes of G and let $\mathrm{Con}^{\#}(G)$ be the set of the conjugacy classes of elements of G of prime power order. For a fixed prime p the conjugacy class of a p-regular element is called p-regular class, and we put

$\mathrm{Con}^{\#}_{p'}(G) = \{C \mid C \in \mathrm{Con}^{\#}(G)$ and C is a p-regular class$\}$

In reference [1], R. Baer characterized all finite groups having the property: $|C|$ is a prime power for each $C \in \mathrm{Con}^{\#}(G)$. D. Chillag and M. Herzog described the structure of G under some assumption on $\mathrm{Con}(G)$[2]. Y. Ninomiya classified finite nonsolvable groups with exactly three p-regular clases[3]. Our main purpose in this note is to improve the following well-known results in reference [2]:

(1) Let p be a prime. $p \nmid |C|$ for each $C \in \mathrm{Con}(G)$ if and only if G has a Sylow p-subgroup in its center.

(2) If $4 \nmid |C|$ for each $C \in \mathrm{Con}(G)$, then G is solvable.

(3) If $|C|$ is a squarefree number for each $C \in \mathrm{Con}(G)$, then G is supersolvable and $dl(G) \leqslant 3$, where $dl(G)$ denotes the derived length of G, and both $|G/F(G)|$ and $|F(G)'|$ are squarefree numbers.

The proofs of our theorems require the following theorem, which is a consequence of the classification of the finite simple groups.

Theorem (FKS)[4] Let G be a transitive permutation group on a set K with $|K| > 1$. Then there exists a prime p and an element $x \in G$ of order a power of p such that x acts without fixed point on K.

Results and Proofs The hypothesis of every theorem of this paper is inherited by normal subgroups and quotient groups by Lemma 1.1 of reference [2], so we can use induction freely in our proofs.

Theorem 1 Let p be a fixed prime. Then $p \nmid |C|$ for each $C \in \mathrm{Con}^{\#}_{p'}(G)$ if and only if G has a Sylow p-subgroup in its center.

Proof If G has a normal subgroup N such that $1 < N < G$, then induction implies that $PN/N \leqslant Z(G/N)$, where $P \in Syl_p(G)$, and that $P \leqslant Z(PN)$. Hence $P \trianglelefteq G$ and P are abelian. Thus the hypothesis implies that $G = P \times O_{p'}(G)$ as required. We therefore may assume that G is a nonabelian simple group.

Let $1 \neq x \in Z(P)$ and $Cl_G(x) = \{x^g \mid g \in G\}$. Then G acts on $Cl_G(x)$ by conjugation and G is a transitive permutation group on $Cl_G(x)$. By FKS-theroem there exists a prime r and element $y \in G$ of order a power of r such that

$$(x^h)^y \neq x^h \quad \forall h \in G.$$

On the other hand, $p \nmid |Cl_G(x)|$ because $x \in Z(P)$, so $r \neq p$ and hence $p \nmid |Cl_G(y)|$ by hypothesis. From this we have $P \leqslant C_G(y^g)$ for some $g \in G$, in particular

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y centrarizes $x^{g^{-1}}$. This is a contradiction and the proof is complete.

Theorem 2 Let p be the smallest prime divisor of $|G|$. If $p^2 \nmid |C|$ for each $C \in \mathrm{Con}_p^{\#}(G)$, then G is p-nilpotent, in particular, G is solvable.

Proof We firstly show that G is not a nonabelian simple group. Suppose that G is. Then by Feit-Thompson's theorem on the solvability of a group of odd order, G has at least one central involution, say u, and $p = 2$. As in the proof of Theorem 1, there is a prime $r \neq p = 2$ and an element $x \in G$ of order a power of r such that $(u^g)^x \neq u^g$ for any $g \in G$. On the other hand, by hypothesis $|G: C_G(x)| \leqslant 2$. Let S be a Sylow 2-subgroup of G such that $u \in Z(S)$ and let T be a Sylow 2-subgroup of $C_G(x)$. We have $T^h \leqslant S$ for some $h \in G$ and obviously $T^h \leqslant C_G(x^h)$. If $u \in T^h$, then $u \in C_G(x^h)$ and so $(u^{h^{-1}})^x = u^{h^{-1}}$. This is a contradiction. If $u \notin T^h$, then $|S: T^h| = 2$. By a lemma of Thompson[5], some conjugate of u, say u^g, lies in T^h. Then $u^g \in C_G(x^h)$ so that $(u^{gh^{-1}})^x = u^{gh^{-1}}$, again a contradiction. The above argument shows that G can not be any nonabelian simple group, and induction implies that G is solvable.

Let M be a maximal subgroup of G and $M \trianglelefteq G$. Then G/M is of order q, where q is a prime. If $q = p$, induction implies that G is p-nilpotent. We therefore may assume that $q \neq p$. Again applying induction we also may assume that $M = P \in Syl_p(G)$. Thus $G = P<x>$, where $P \trianglelefteq G$ and $|x| = q \neq p$. By hypothesis,

$$p^2 \nmid |G: C_G(x)| = |P: C_P(x)|$$

so

$$|G: C_G(x)| = 1, \text{ or } p$$

which implies that $C_G(x) \trianglelefteq G$ and hence $<x> \trianglelefteq G$. In particular, G is p-nilpotent. This completes the proof.

Corollary 3 If $4 \nmid |C|$ for each $C \in \mathrm{Con}_2^{\#}(G)$, then G is 2-nilpotent.

Lemma 4 Let p be a prime. If $p^2 \nmid |C|$ for each $C \in \mathrm{Con}_p^{\#}(G)$ and if G' is nilpotent, then $P/O_p(G)$ is an elementary abelian p-group, where $P \in Syl_p(G)$.

Proof By induction we may assume that $O_p(G) = 1$. Hence $G' \leqslant F(G) \leqslant O_{p'}(G)$ and $G = PO_{p'}(G)$, where $P \in Syl_p(G)$. Again applying induction we also may assume that $G = PF(G)$. Put $\bar{G} = G/\Phi(G)$ and $H/\Phi(G) = O_p(\bar{G})$. Then $H = Q\Phi(G) \trianglelefteq G$, where $Q \in Syl_p(H)$. We have $G = N_G(Q)\Phi(G) = N_G(Q)$. Hence $Q \leqslant O_p(G) = 1$, namely $O_p(\bar{G}) = 1$ and $p \nmid |\Phi(G)|$. By induction we may assume that $\Phi(G) = 1$. This implies that $F(G) = N_1 \times \cdots \times N_r$ is a direct product of elementary abelian groups N_i. Any $x \in \bigcup_{i=1}^{r} N_i$ is of order a prime and x is p-regular. Noting $G = PF(G)$ we have $|P: C_P(x)| \leqslant p$ by hypothesis. Thus

$$\Phi(P) \leqslant \bigcap_{x \in F(G)} C_P(x) = C_P(F(G)) = 1$$

which implies that P is an elementary abelian p-group. This completes the proof.

Theorem 5 If for any prime p and any $C \in \mathrm{Con}_p^{\#}(G)$ $p^2 \nmid |C|$, then G is supersolvable and $G/F(G)$ is a direct product of elementary abelian groups.

Proof If G contains a normal subgroup of prime order, then induction implies that G is supersolvable. Thus we may assume that G contains no normal subgroup of prime order. By Theorem 2, G is solvable. Let N be a minimal normal subgroup of G. Then N is an elementary abelian group of order p^n for some prime p and an integer $n \geqslant 2$. If N is not contained in some maximal subgroup M of G, then $G = MN$ is a semi-direct product. Then $M \cong G/N$ is supersolvable and hence M contains a normal subgroup Q of prime order. Set $Q = <x>$. As Q is not normal in G, $M = N_G(Q)$ and therefore $C_G(x) \cap N = 1$. Consequently $p^n \mid |NC_G(x): C_G(x)| \mid |G: C_G(x)|$. On the other hand, $M = N_G(Q)$ also implies that x is a p'-element, by hypothesis $p^2 \nmid |G: C_G(x)|$. This is a contradiction. Thus N is contained in every maximal subgroup of G, so that the Frattini subgroup $\Phi(G)$ of G is nontrivial and by induction $G/\Phi(G)$ is supersolvable, and hence G is supersolvable. Other conclusion of the theorem follows from Lemma 4.

Corollary 6 If $|C|$ is a squarefree number for each $C \in \mathrm{Con}^{\#}(G)$, then G is supersolvable, $dl(G) \leqslant 3$, $G/F(G)$ is a direct-product of elementary abelian groups and $|F(G)'|$ is a squarefree number.

Proof By Theorem 5 we need only show that $dl(G) \leqslant 3$ and $|F(G)'|$ is squarefree number. We have $G' \leqslant F(G)$. As $F(G)$ satisfies the hypothesis of the theorem, by reference [2], $|F(G)'|$ is squarefree number. Hence $G''' = 1$, namely $dl(G) \leqslant 3$.

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